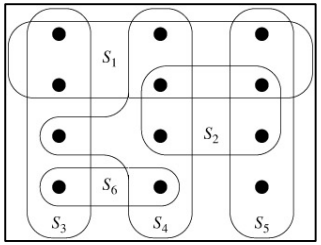
Basics III: Dynamic Programming, Integer Linear Programming

Recent Advances in Parameterized Complexity, 2017

Meirav Zehavi

Outline

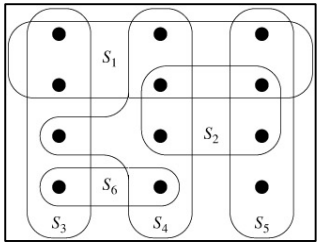
1. Dynamic Programming
   i. Set Cover
   ii. Steiner Tree
2. Integer Linear Programming
   i. Closest String
   ii. Imbalance



DP: Set Cover

Set Cover. Given a universe U, and a family F of subsets of U, what is the minimum size of a subfamily F' of F such that $\cup F' = U$?

Parameter. $n = |U|$.



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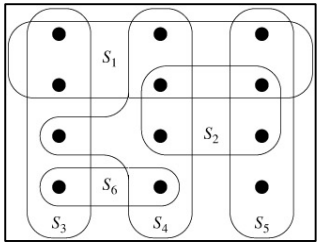
Denote $F = \{S_1, S_2, \ldots, S_m\}$.



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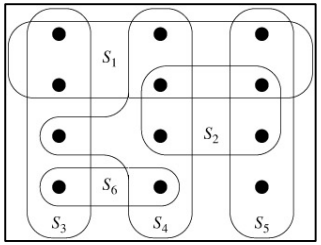
DP Table. Entry M[i,U'] for all $i \in [m]_0$ and $U' \subseteq U$.



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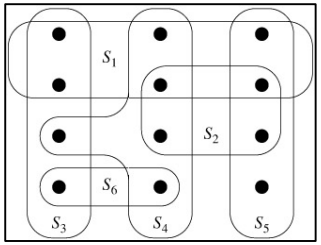

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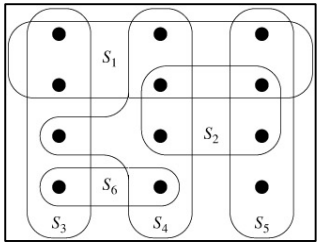

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Recursion. M[i,U'] = min{M[i-1,U'],
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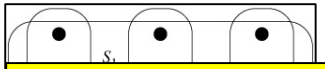

The size of the smallest subfamily F' of $\{S_1,S_2,\ldots,S_i\}$ such that $U' \subseteq F'$ might be smaller that the size of the smallest subfamily F' of $\{S_1,S_2,\ldots,S_i\}$ such that $U' = F'$.

Meaning. M[i,U']: What is the minimum size of a subfamily F' of $\{S_1,S_2,\ldots,S_i\}$ such that $U' \subseteq \cup F'$?

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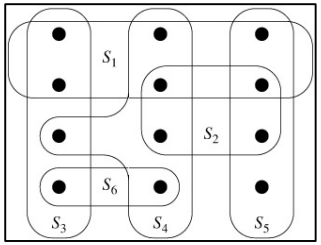

DP: Set Cover

Running time. $O(2^n m)$.

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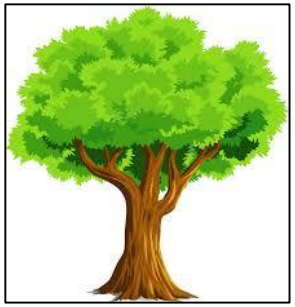

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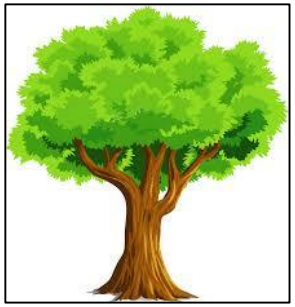
Steiner Tree. Given a graph G and a terminal set $W \subseteq V(G)$, what is the minimum size of a subtree T of G such that $W \subseteq V(G)$?

Parameter. $k = |W|$.



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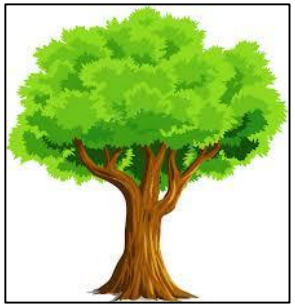
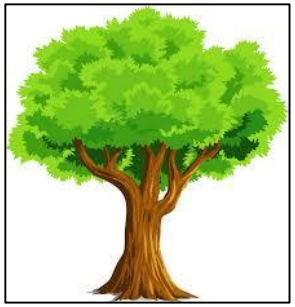
(Size of subtree $T = |E(T)|$.)



DP: Steiner Tree

Assumptions.

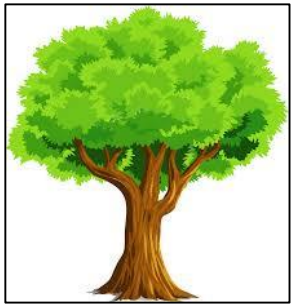
1. $|W| > 1$.



DP: Steiner Tree

Assumptions.

1. $|W| > 1$.
2. G is connected.

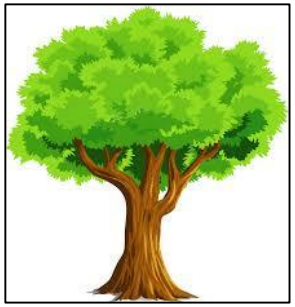


DP: Steiner Tree

Assumptions.

1. **$|W| > 1$.**
2. ***G* is connected.**

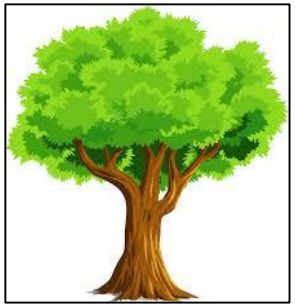
dist(u,v) = length of a shortest path in G between u and v.



DP: Steiner Tree

Assumptions.

1. $|W| > 1$.
2. G is connected.
3. W is an independent set, and each vertex in W is a leaf in G.



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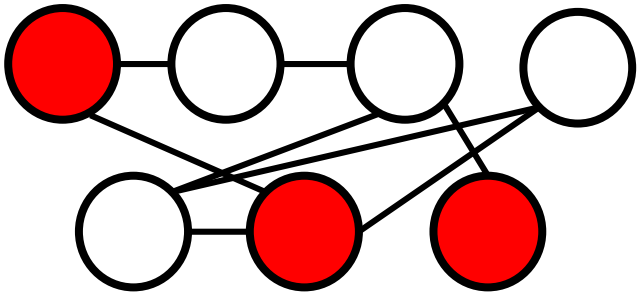

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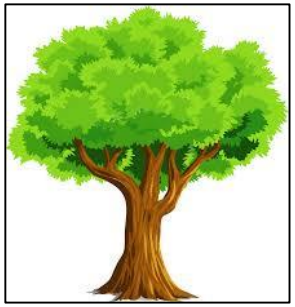

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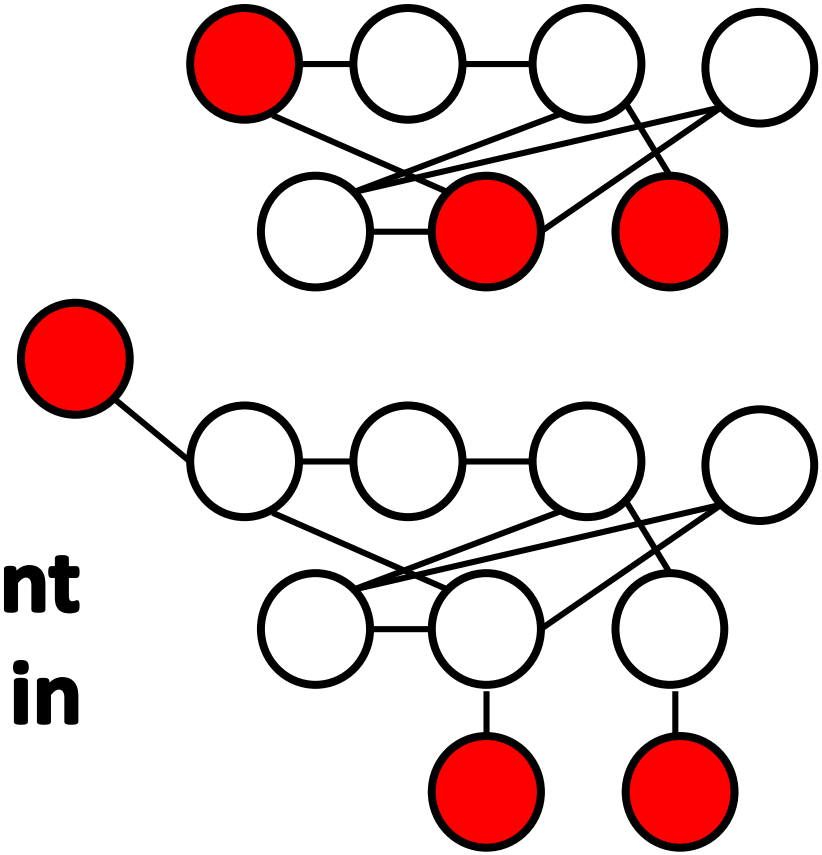

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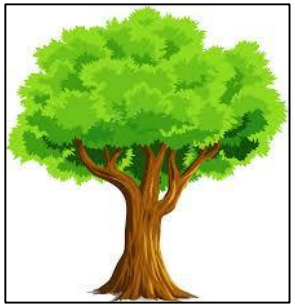

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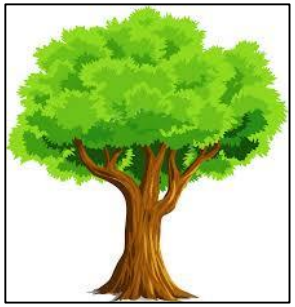

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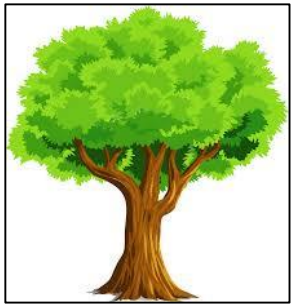

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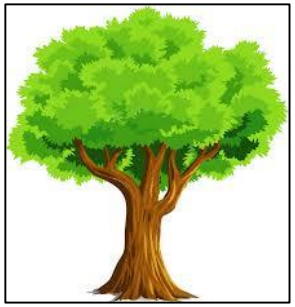
Basis. $M[v,\{w\}] = \text{dist}(v,w)$.



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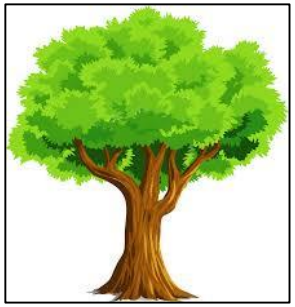
Recursion. $M[v,W'] =$

$$\min_{\substack{u \in V(G) \setminus W \\ X \subset W',\ X \neq \emptyset}} \{M[u,X] + M[u,W' \setminus X] + \mathrm{dist}(v,u)\}.$$



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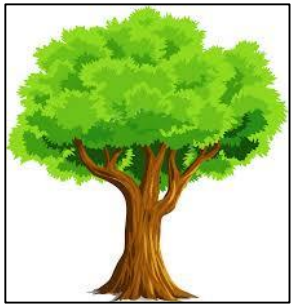
Proof. $M[v,W'] \leq \min_{\substack{u \in V(G) \setminus W \\ X \subset W',\, X \neq \emptyset}} \{M[u,X] + M[u,W' \setminus X] + \mathrm{dist}(v,u)\}.$



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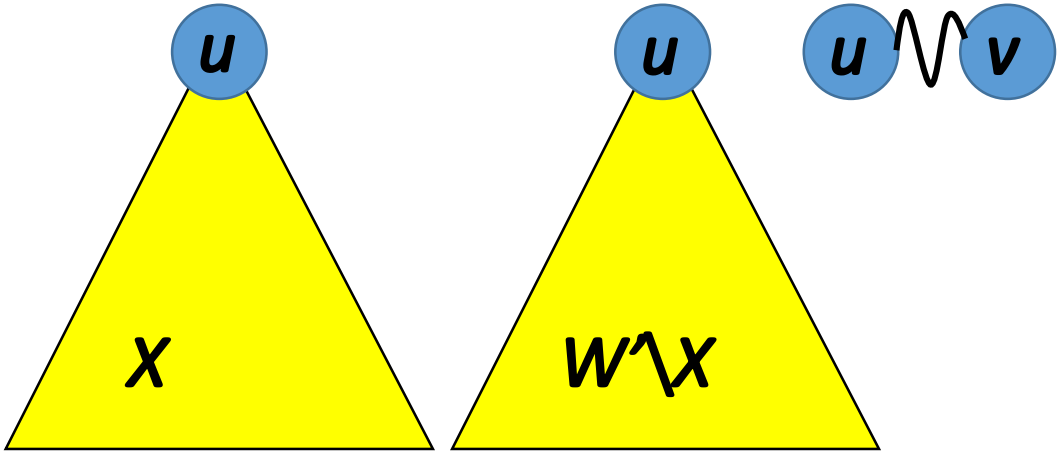

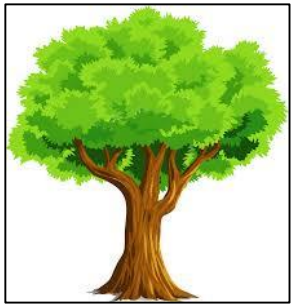

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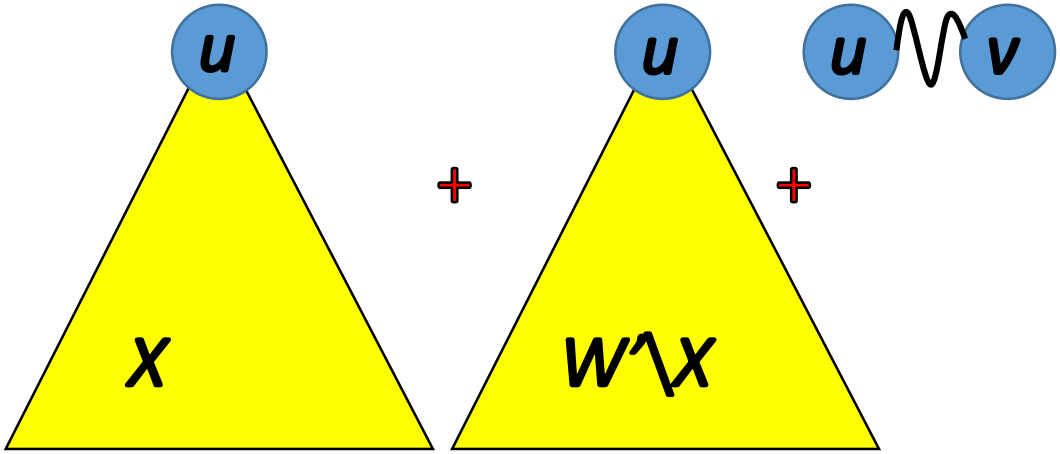
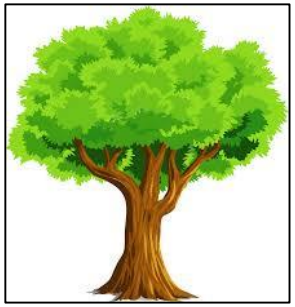


Subgraph S with at most * edges such that $W' \cup \{v\} \subseteq V(S)$.



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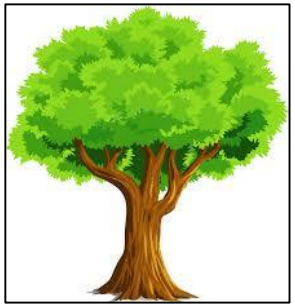

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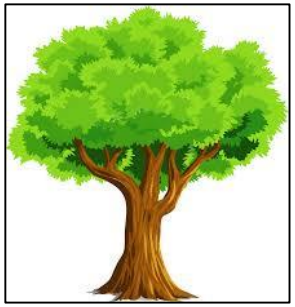

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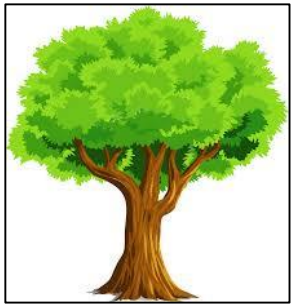
$|W'|>1$,

W is an independent set of leaves



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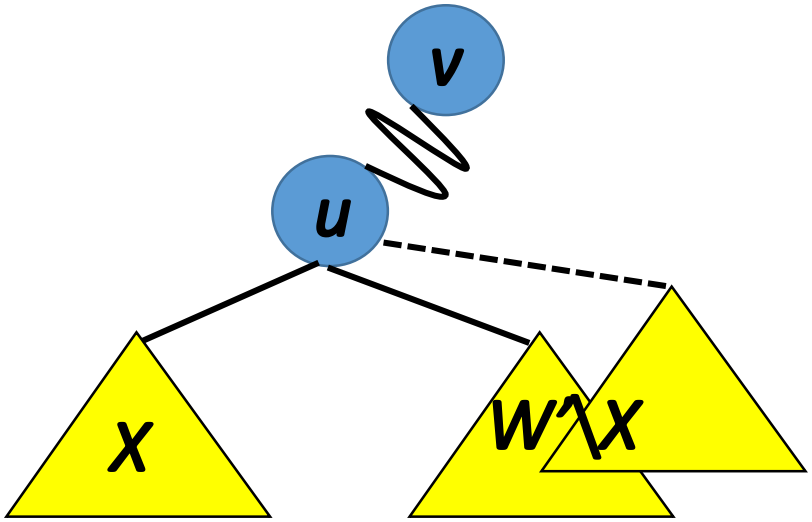

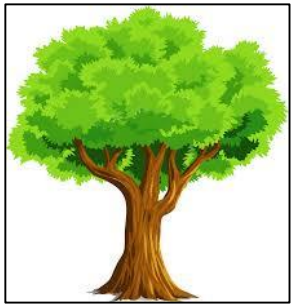

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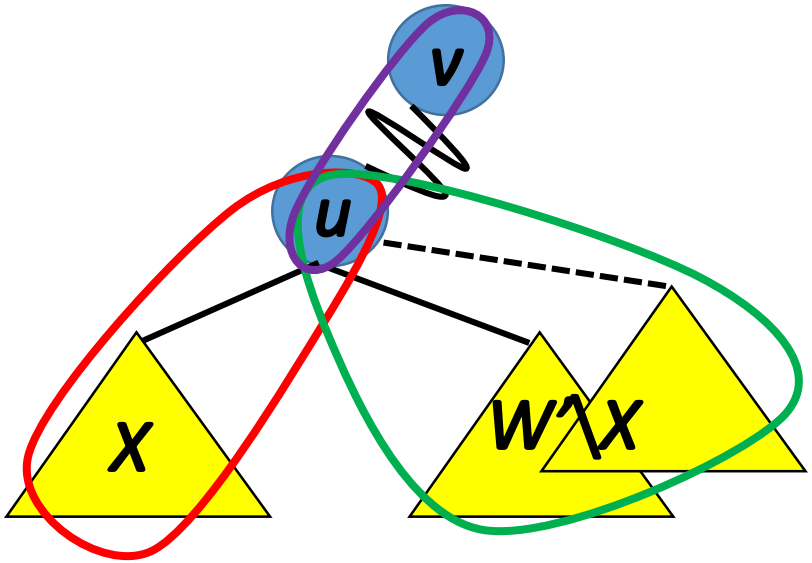

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$$\begin{array}{l} a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,p}x_p \leq b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,p}x_p \leq b_2 \\ \vdots \quad \vdots \quad \ddots \quad \vdots \quad \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \ldots + a_{m,p}x_p \leq b_m \end{array}$$

ILP: Introduction

Form.

$$\begin{array}{l} a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,p}x_p \leq b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,p}x_p \leq b_2 \\ \vdots \quad \vdots \quad \ddots \quad \vdots \quad \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \ldots + a_{m,p}x_p \leq b_m \end{array}$$

Is there an assignment of integers to variables that satisfies all inequalities?

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ILP is solvable in $p^{O(p)}L$ arithmetic operations. [L – input size.] (Lenstra 83, Kannan 87)

$$\begin{array}{ccccc} a_{1,1}x_1 + a_{1,2}x_2 & + \ldots + & a_{1,p}x_p & \leq b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 & + \ldots + & a_{2,p}x_p & \leq b_2 \\ \vdots \quad \vdots & \ddots & \vdots & \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 & + \ldots + & a_{m,p}x_p & \leq b_m \end{array}$$

ILP: Closest String

Closest String. Given k strings $s_1,\ldots,s_k$ over alphabet $[n]$, each of length L, and a positive integer d, decide if there is a string s^* s.t.

$H(s^*,s_i)$:= Hamming-dist$(s^*,s_i) \leq d$ for all $i \in [k]$.

$$\begin{array}{ccccc} a_{1,1}x_1 + a_{1,2}x_2 & + \ldots + & a_{1,p}x_p & \leq b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 & + \ldots + & a_{2,p}x_p & \leq b_2 \\ \vdots \quad \vdots & \ddots & \vdots & \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 & + \ldots + & a_{m,p}x_p & \leq b_m \end{array}$$

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s^*	?	?	?	?	?		?
s_1	3	5	1	9	5		9
s_2	6	1	1	8	6		9
s_k	8	5	1	1	5		4

$$\begin{array}{l} a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,p}x_p \le b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,p}x_p \le b_2 \\ \vdots \qquad \vdots \qquad \ddots \qquad \vdots \qquad \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \ldots + a_{m,p}x_p \le b_m \end{array}$$

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ILP: Closest String

Column type. The type of a column *c* is the column vector associated with it.

s^*	?	?	?	?	?		?
s_1	3	5	1	9	5		9
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Column type. The type of a column *c* is the column vector associated with it.

- k^k types.

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s_1	3	5	1	9	5		9
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ILP: Closest String

Column type. The type of a column *c* is the column vector associated with it.

- k^k types.
- n_t = the number of columns of type *t*.

s^*	?	?	?	?	?		?
s_1	3	5	1	9	5		9
s_2	6	1	1	8	6		9
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ILP: Closest String

Variables. x_a^t **for every type** ***t*** **and letter** ***a*****.**

$$\begin{array}{ccccc} a_{1,1}x_1 + a_{1,2}x_2 & + \ldots + & a_{1,p}x_p & \leq b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 & + \ldots + & a_{2,p}x_p & \leq b_2 \\ \vdots \quad \vdots & \ddots & \vdots & \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 & + \ldots + & a_{m,p}x_p & \leq b_m \end{array}$$

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- $k^k k$ **variables.**

$$\begin{array}{l} a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,p}x_p \leq b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,p}x_p \leq b_2 \\ \vdots \qquad \vdots \qquad \ddots \qquad \vdots \qquad \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \ldots + a_{m,p}x_p \leq b_m \end{array}$$

ILP: Closest String

Variables. x_a^t **for every type *t* and letter *a*.**

- $k^k k$ **variables.**
- x_a^t**: # occurrences of *a* in columns of type *t* in *s**.**

$$\begin{array}{l} a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,p}x_p \leq b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,p}x_p \leq b_2 \\ \vdots \qquad \vdots \qquad \ddots \qquad \vdots \qquad \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \ldots + a_{m,p}x_p \leq b_m \end{array}$$

ILP: Closest String

Variables. x_a^t for every type t and letter a.

- $k^k k$ variables.
- x_a^t: # occurrences of a in columns of type t in s^*.

Constraints. $\forall t \in [k^k]$ and $a \in [k]$: $x_a^t \geq 0$.

$$\begin{array}{lllll} a_{1,1}x_1 + a_{1,2}x_2 & + \ldots + a_{1,p}x_p & \leq b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 & + \ldots + a_{2,p}x_p & \leq b_2 \\ \vdots \quad \vdots & \ddots \quad \vdots & \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 & + \ldots + a_{m,p}x_p & \leq b_m \end{array}$$

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$\forall i \in [k]$: $\sum_{t \in [k^k]} \sum_{a \in [k]} x_a^t \leq d$.

a is not the letter of s_i in a column of type t

Running time. $O^*(2^{k^{O(k)}})$.

Variables. x_a^t for every type t and letter a.

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a is not the letter of s_i in a column of type t

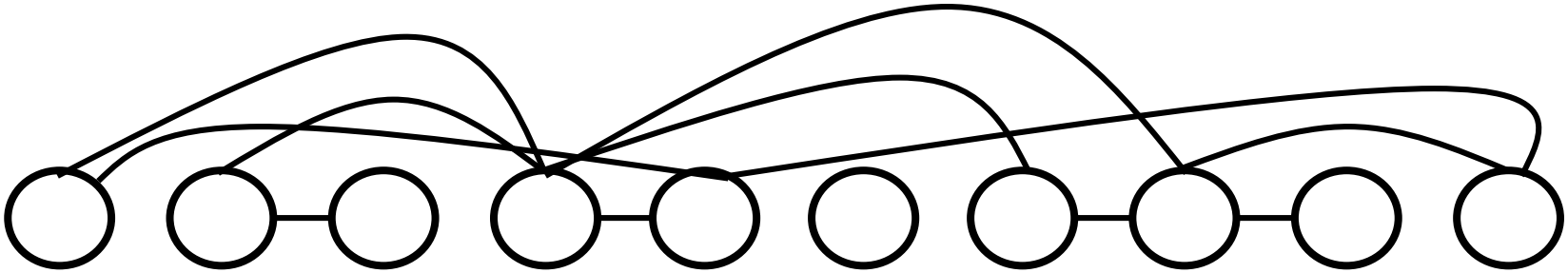
Outline

1. Dynamic Programming
   i. Set Cover
   ii. Steiner Tree
2. Integer Linear Programming
   i. Closest String
   ii. Imbalance

$$\begin{array}{ccccc} a_{1,1}x_1 + a_{1,2}x_2 & + \ldots + & a_{1,p}x_p & \leq b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 & + \ldots + & a_{2,p}x_p & \leq b_2 \\ \vdots \quad \vdots & \ddots & \vdots & \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 & + \ldots + & a_{m,p}x_p & \leq b_m \end{array}$$

ILP: Imbalance

Let *G* be a graph, and *π* be an ordering of *V*(*G*).



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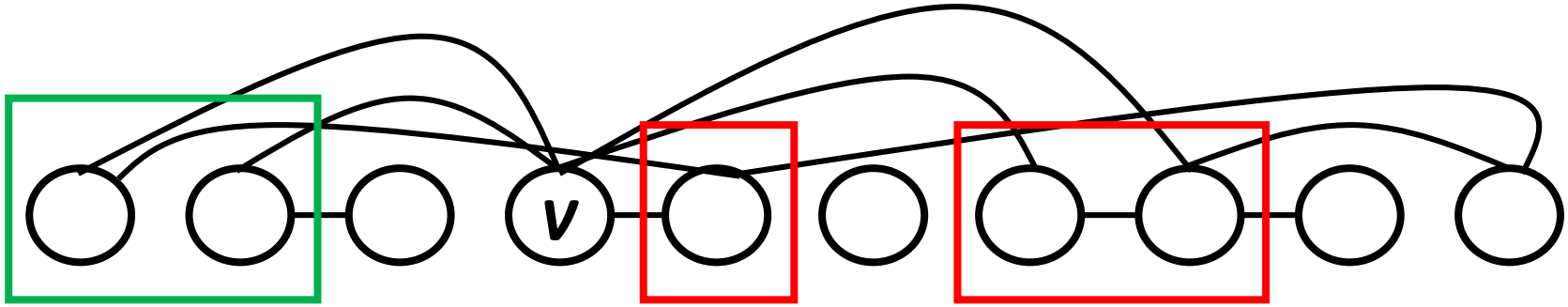
ILP: Imbalance

Let G be a graph, and π be an ordering of $V(G)$.

$L_\pi(v) = \{u \in N(v) : \pi(u) < \pi(v)\}$, $R_\pi(v) = \{u \in N(v) : \pi(u) > \pi(v)\}$.



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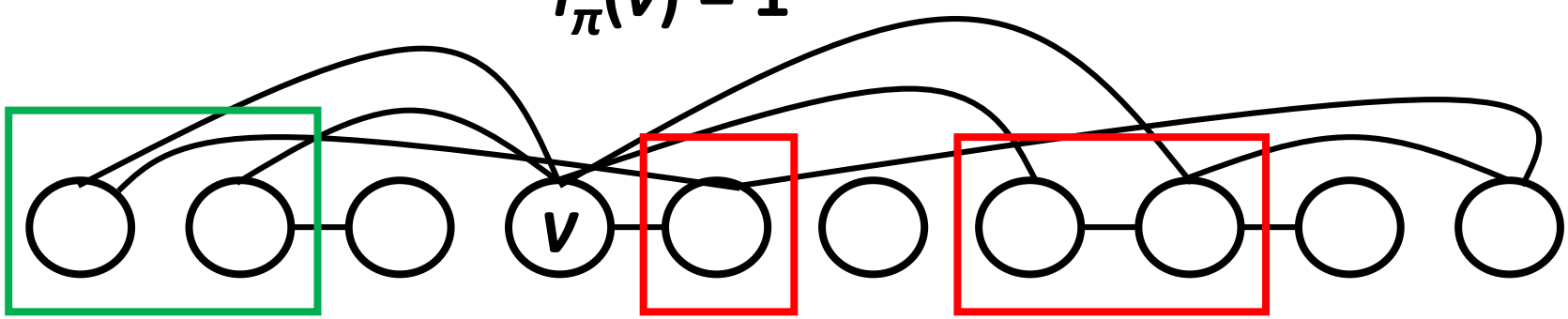
Imbalance of v in π. $i_\pi(v) = ||L_\pi(v)| - |R_\pi(v)||$.

$i_\pi(v) = 1$



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Imbalance of v in π. $i_\pi(v) = ||L_\pi(v)| - |R_\pi(v)||$.

Imbalance of π. $i_\pi = \sum_{v \in V(G)} i_\pi(v)$.

$$\begin{array}{ccccc} a_{1,1}x_1 + a_{1,2}x_2 & + \ldots + & a_{1,p}x_p & \leq b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 & + \ldots + & a_{2,p}x_p & \leq b_2 \\ \vdots \quad \vdots & \ddots & \vdots & \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 & + \ldots + & a_{m,p}x_p & \leq b_m \end{array}$$

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Imbalance of v in π. $i_\pi(v) = ||L_\pi(v)| - |R_\pi(v)||$.

Imbalance of π. $i_\pi = \sum_{v \in V(G)} i_\pi(v)$.

Imbalance. Given a graph G, decide if there is an ordering π of $V(G)$ of imbalance at most d.

$$\begin{array}{ccccc} a_{1,1}x_1 + a_{1,2}x_2 & + \ldots + & a_{1,p}x_p & \leq b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 & + \ldots + & a_{2,p}x_p & \leq b_2 \\ \vdots \quad \vdots & \ddots & \vdots & \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 & + \ldots + & a_{m,p}x_p & \leq b_m \end{array}$$

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Imbalance. Given a graph G, decide if there is an ordering π of $V(G)$ of imbalance at most d.

Parameter. The size k of a vertex cover X of G.

$$\begin{array}{l} a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,p}x_p \leq b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,p}x_p \leq b_2 \\ \vdots \quad \vdots \quad \ddots \quad \vdots \quad \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \ldots + a_{m,p}x_p \leq b_m \end{array}$$

ILP: Imbalance

For every possible ordering π_X of X, what is the minimum imbalance of an ordering π of $V(G)$ that agrees with π_X?

$$\begin{array}{ccccc} a_{1,1}x_1 + a_{1,2}x_2 & + \ldots + & a_{1,p}x_p & \leq b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 & + \ldots + & a_{2,p}x_p & \leq b_2 \\ \vdots \quad \vdots & \ddots & \vdots & \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 & + \ldots + & a_{m,p}x_p & \leq b_m \end{array}$$

ILP: Imbalance

For every possible ordering π_X of X, what is the minimum imbalance of an ordering π of $V(G)$ that agrees with π_X?

$k!$ options

$$\begin{array}{l} a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,p}x_p \leq b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,p}x_p \leq b_2 \\ \vdots \quad \vdots \quad \ddots \quad \vdots \quad \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \ldots + a_{m,p}x_p \leq b_m \end{array}$$

ILP: Imbalance

For every possible ordering π_X of X, what is the minimum imbalance of an ordering π of $V(G)$ that agrees with π_X?

Denote $X=\{u_1,u_2,\ldots,u_k\}$ and $I = V(G)\setminus X$.

Vertex Type. The type of $v \in I$ is $N(v) \subseteq X$.

$$\begin{array}{ccccc} a_{1,1}x_1 + a_{1,2}x_2 & + \ldots + & a_{1,p}x_p & \leq b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 & + \ldots + & a_{2,p}x_p & \leq b_2 \\ \vdots \quad \vdots & \ddots & \vdots & \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 & + \ldots + & a_{m,p}x_p & \leq b_m \end{array}$$

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Denote $X=\{u_1,u_2,\ldots,u_k\}$ and $I = V(G)\setminus X$.

Vertex Type. The type of $v \in I$ is $N(v) \subseteq X$.

n_S = the number of vertices (in I) of type S.

$$\begin{array}{ccccc} a_{1,1}x_1 + a_{1,2}x_2 & + \ldots + & a_{1,p}x_p & \leq b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 & + \ldots + & a_{2,p}x_p & \leq b_2 \\ \vdots \quad \vdots & \ddots & \vdots & \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 & + \ldots + & a_{m,p}x_p & \leq b_m \end{array}$$

ILP: Imbalance

A vertex *v* is at position $p \in [k]_0$ if $\pi(u_p) < \pi(v) < \pi(u_{p+1})$.

$$\begin{array}{ccccc} a_{1,1}x_1 + a_{1,2}x_2 & + \ldots + & a_{1,p}x_p & \leq b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 & + \ldots + & a_{2,p}x_p & \leq b_2 \\ \vdots \quad \vdots & \ddots & \vdots & \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 & + \ldots + & a_{m,p}x_p & \leq b_m \end{array}$$

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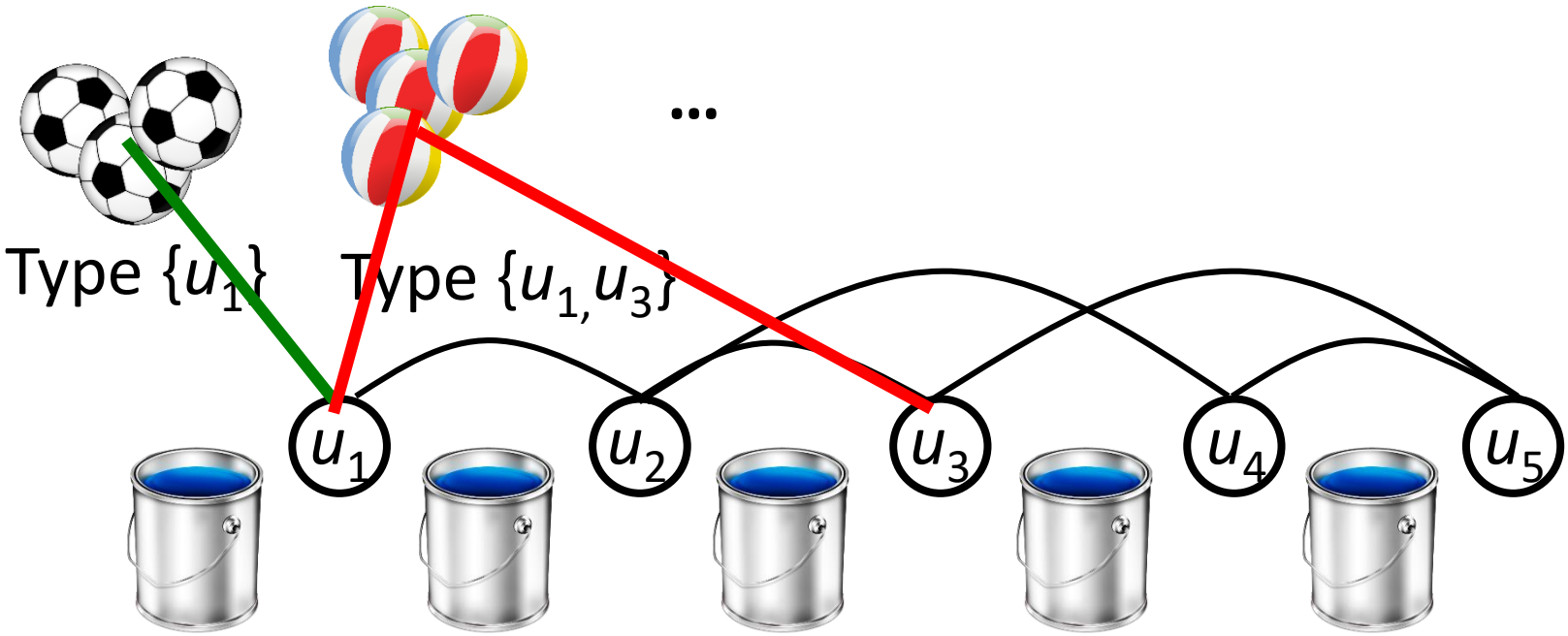

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ILP: Imbalance

Variables. x_p^S **for every type** S **and position** p**.**

- $2^k(k+1)$ **variables.**

$$\begin{array}{ccccc} a_{1,1}x_1 + a_{1,2}x_2 & + \ldots + & a_{1,p}x_p & \leq b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 & + \ldots + & a_{2,p}x_p & \leq b_2 \\ \vdots \quad \vdots & \ddots & \vdots & \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 & + \ldots + & a_{m,p}x_p & \leq b_m \end{array}$$

ILP: Imbalance

Variables. x_p^S **for every type** ***S*** **and position** ***p*****.**

- $2^k(k+1)$ **variables.**
- x_p^S**: # occurrences of** ***S*****-type vertices in position** ***p*** **in** π**.**

$$\begin{array}{ccccc} a_{1,1}x_1 + a_{1,2}x_2 & + \ldots + & a_{1,p}x_p & \leq b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 & + \ldots + & a_{2,p}x_p & \leq b_2 \\ \vdots \quad \vdots & \ddots & \vdots & \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 & + \ldots + & a_{m,p}x_p & \leq b_m \end{array}$$

ILP: Imbalance

Variables. x_p^S for every type S and position p.

- $2^k(k+1)$ variables.
- x_p^S: # occurrences of S-type vertices in position p in π.

Constraints. $\forall S \subseteq X$ and $p \in [k]_0$: $x_p^S \geq 0$.

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$\forall i \in [k]$: $y_i \geq |c_i + \sum_{\{u_i\} \subseteq S \subseteq X} (\sum_{p=0}^{i-1} x_p^S - \sum_{p=i}^{k} x_p^S)|$.

$|N(u_i) \cap \{u_1, \ldots, u_{i-1}\}| - |N(u_i) \cap \{u_{i+1}, \ldots, u_k\}|$

$$\begin{array}{ccccc} a_{1,1}x_1 + a_{1,2}x_2 & + \ldots + & a_{1,p}x_p & \leq b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 & + \ldots + & a_{2,p}x_p & \leq b_2 \\ \vdots \quad \vdots & \ddots & \vdots & \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 & + \ldots + & a_{m,p}x_p & \leq b_m \end{array}$$

ILP: Imbalance

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$\sum_{i=1}^{k} y_i + \sum_{p=0}^{k} \sum_{S \subseteq X} c_p^S x_p^S \leq d$.

$||S \cap \{u_1, \ldots, u_p\}| - |S \cap \{u_{p+1}, \ldots, u_k\}||$

Questions?