Lower bounds under the Exponential Time Hypothesis

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- **Goal**: Fine-grained complexity theory.

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- For every q you can do $(2 - \varepsilon_q)^n$ for qSAT, but $\lim_{q\to\infty} \varepsilon_q = 0$.

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ETH and SETH, first attempt

ETH: $3\mathrm{SAT}$ cannot be solved in time $2^{o(n)}$.
SETH: $\mathrm{CNF\text{-}SAT}$ cannot be solved in time $\mathcal{O}(\alpha^n)$ for any $\alpha < 2$.

Actual statements

$$\delta_q = \inf\{c : \text{ There is an } \mathcal{O}(2^{cn}) \text{ algorithm for } q\mathrm{SAT}\}$$

Exponential Time Hypothesis, ETH

$$\delta_3 > 0.$$

There is a $c > 0$ such that $3\mathrm{SAT}$ cannot be solved in $\mathcal{O}(2^{cn})$ time.

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- ETH is widely believed, SETH is disputed.

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Vertex Cover

I: graph G and $k \in \mathbb{N}$

Q: Is there a set $X \subseteq V(G)$ with $|X| \leqslant k$ such that every edge of G has at least one endpoint in X?

Vertex Cover reduction

Let us inspect the standard reduction from 3SAT to Vertex Cover.

clauses

variables

Variable gadgets $\rightsquigarrow$ Edges

Clause gadgets $\rightsquigarrow$ Triangles

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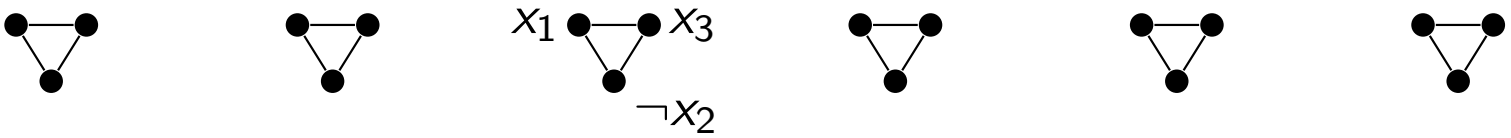

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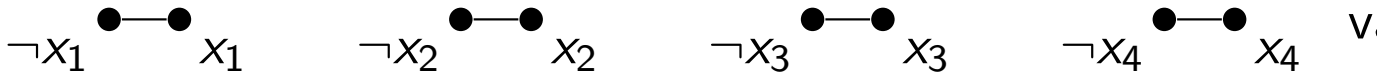

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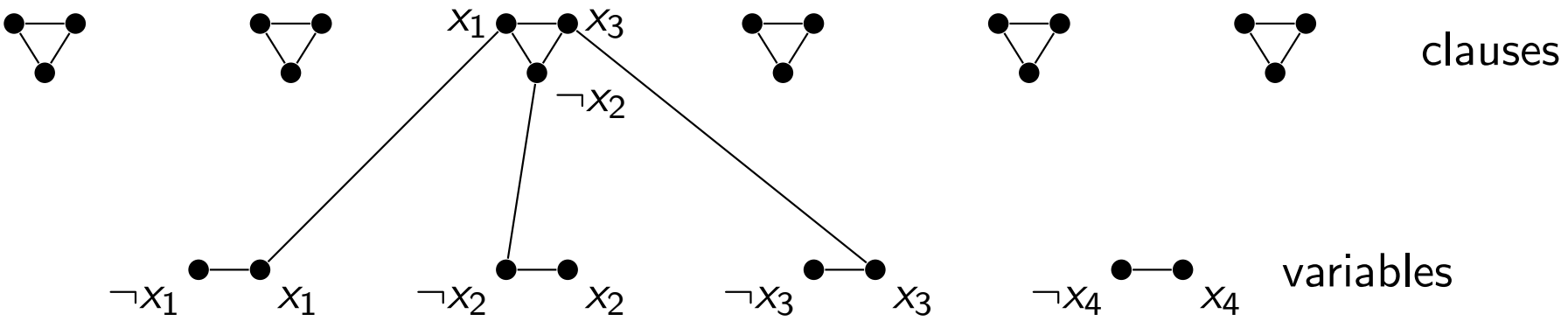

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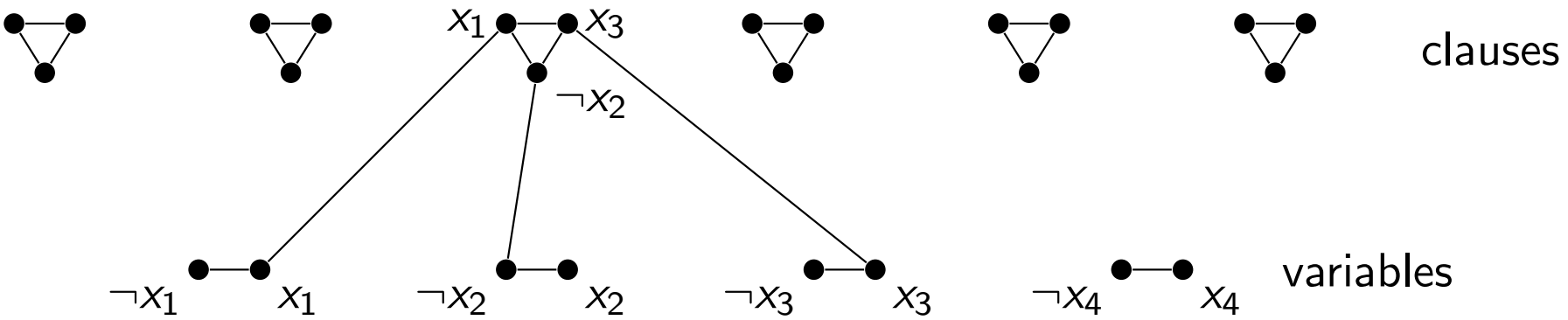

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φ is satisfiable iff the created graph has a vertex cover of size $n + 2m$.

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- **Gap**: between upper bound $2^{\mathcal{O}(N)}$ and lower bound $2^{o(N^{1/3})}$.
- If we started with an instance of $3\text{-}\mathrm{SAT}$ that was **sparse**, that is, $m = \mathcal{O}(n)$, then a $2^{o(N)}$ lower bound would follow.

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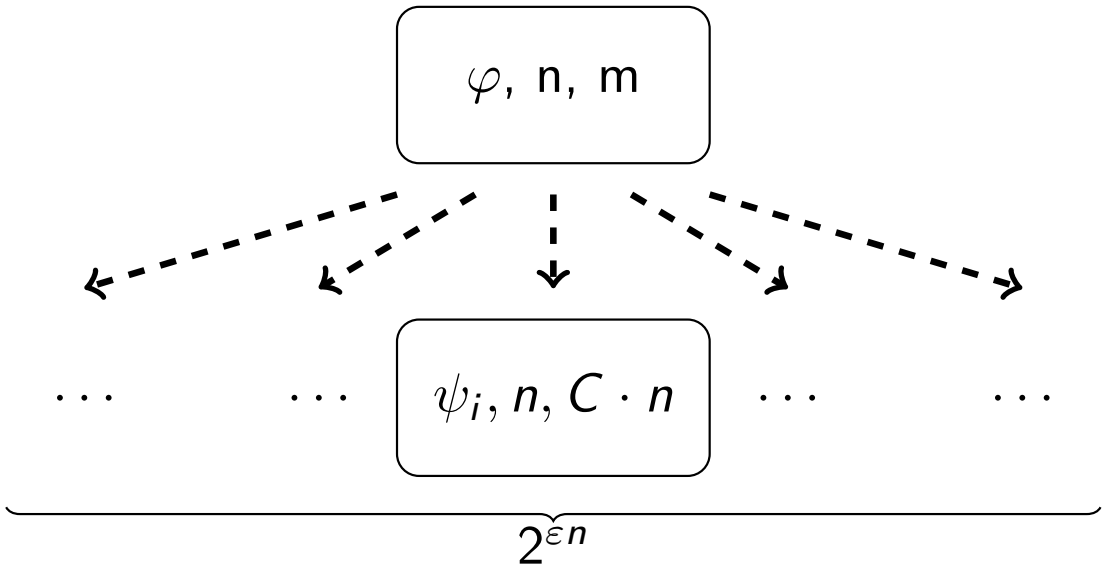

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- Use Sparsification Lemma for $\varepsilon = d/2$. Denote $C = C(3, \varepsilon)$.

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- Consider any $d > 0$. We want to solve $3\mathrm{SAT}$ in time $\mathcal{O}^\star(2^{dn})$.
- Use Sparsification Lemma for $\varepsilon = d/2$. Denote $C = C(3, \varepsilon)$.
- Solve each ψ_i by $\mathcal{A}_{c'}$, where $c' = \frac{d}{2(C+1)}$.

Sparsification Lemma: corollary

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Unless ETH fails, there is a constant $c > 0$, such that no algorithm solves $3\mathrm{SAT}$ in time $\mathcal{O}^\star(2^{c(n+m)})$.

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Corollary

Under ETH, there is no $2^{o(n+m)}$-time algorithm for $3\mathrm{SAT}$, so also no $2^{o(N+M)}$-time algorithm for Vertex Cover.

Transfer of hardness

Hardness under ETH via reductions

Suppose there is a reduction from $3\mathrm{SAT}$ to a problem L that produces an instance of size $N \leqslant f(n+m)$.

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- Let's see tight bounds for another form of running time.

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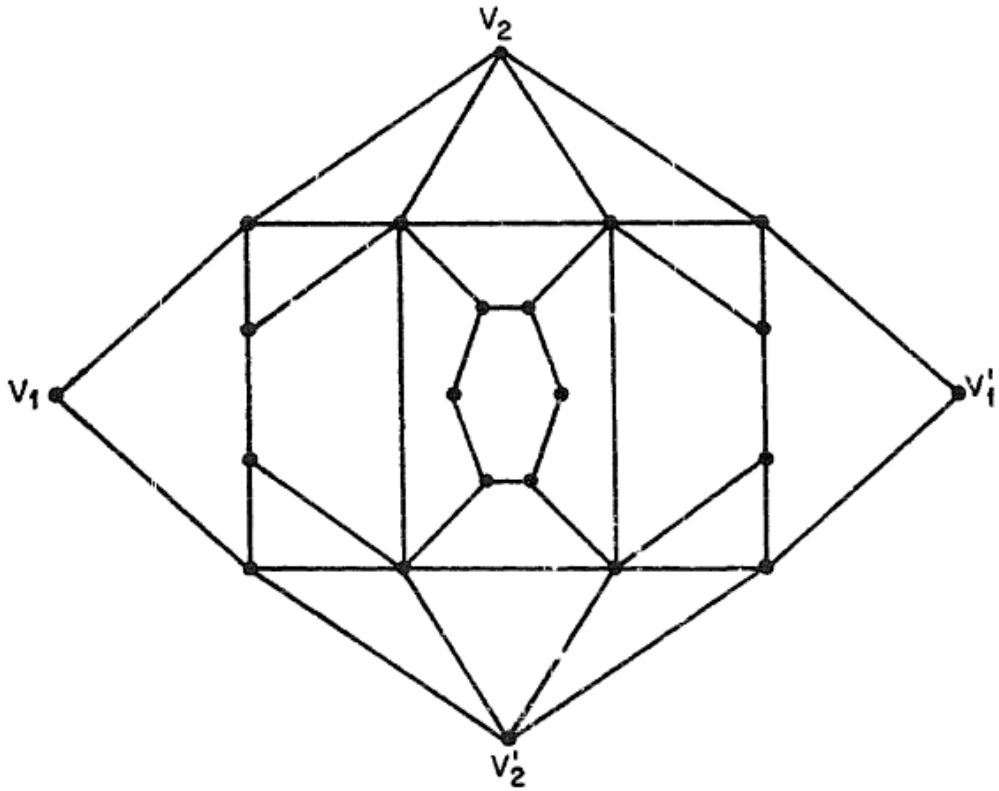

Fig. 11. Crossover H for Theorem 2.7.

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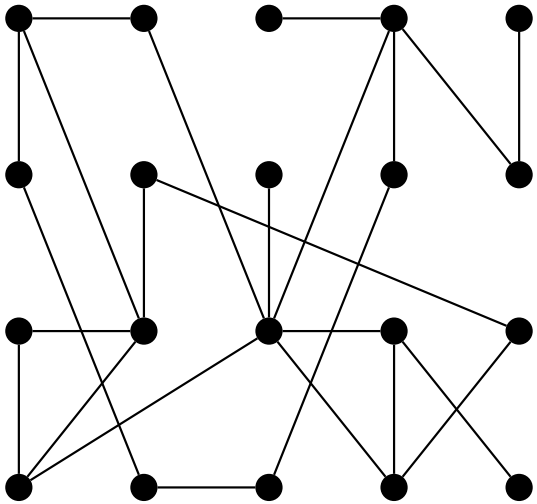
- We start from an instance of 3-Coloring, for which we already have a $2^{o(n)}$ lower bound.

ETH hardness for Clique



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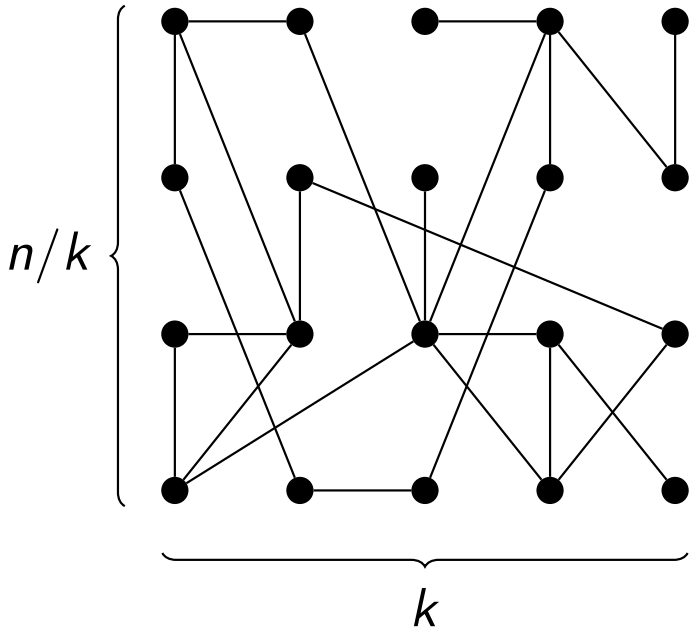

Partition vertices into k groups with n/k vertices each.

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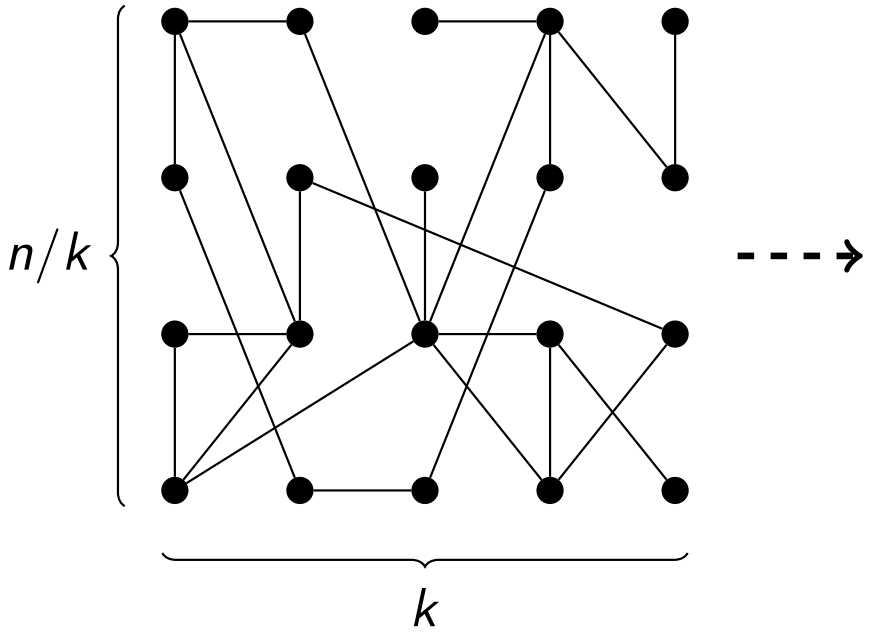

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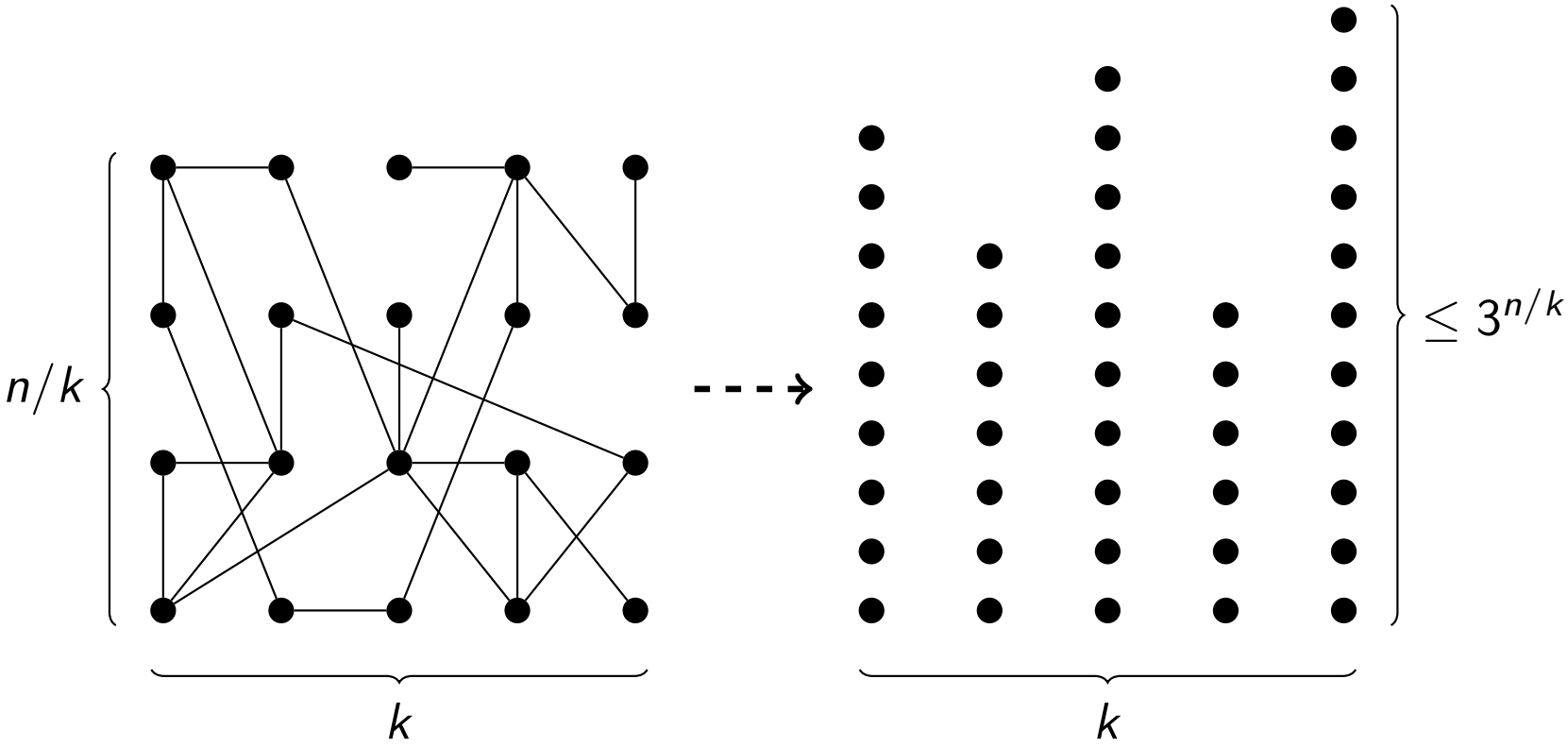

Create one vertex per each **consistent coloring** of each group.

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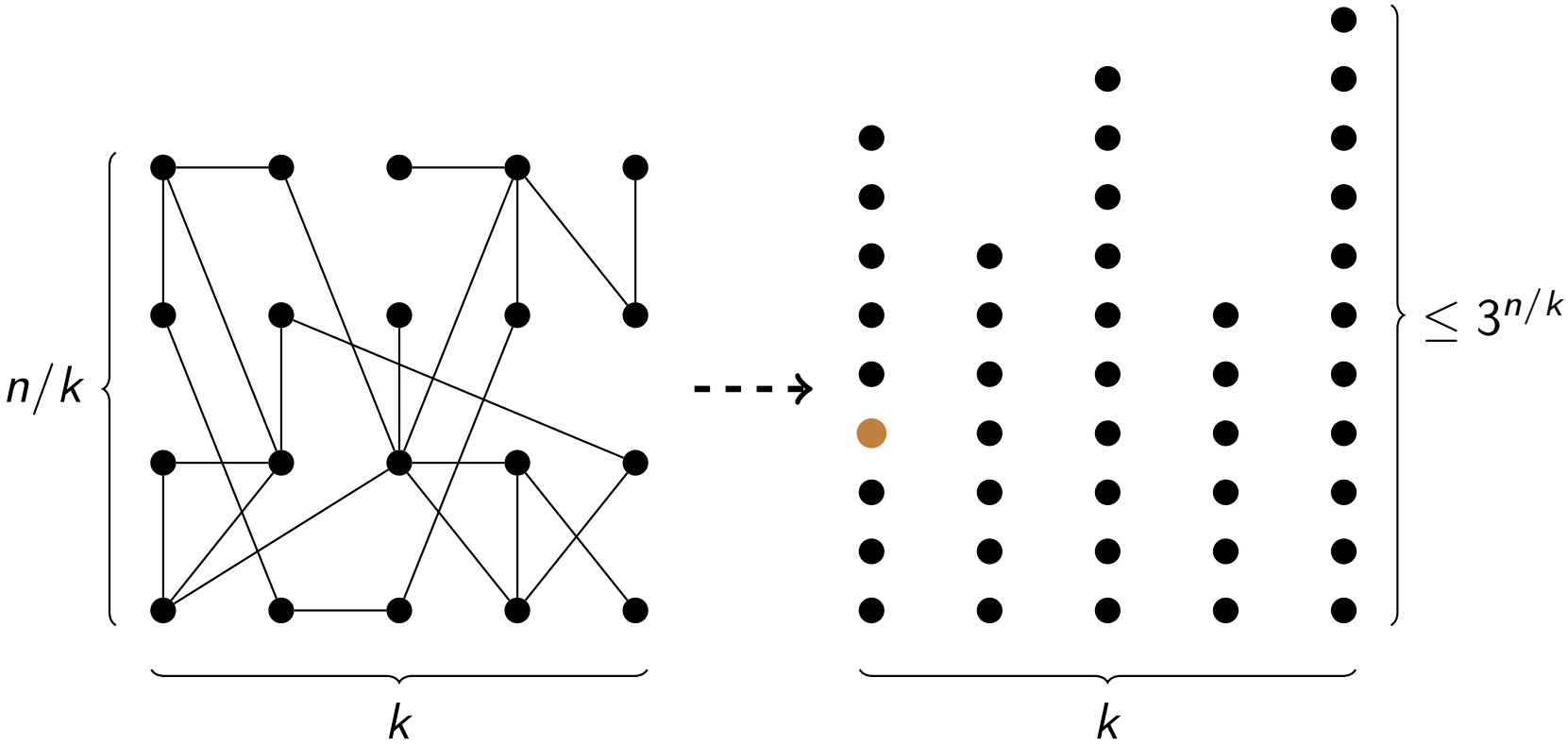

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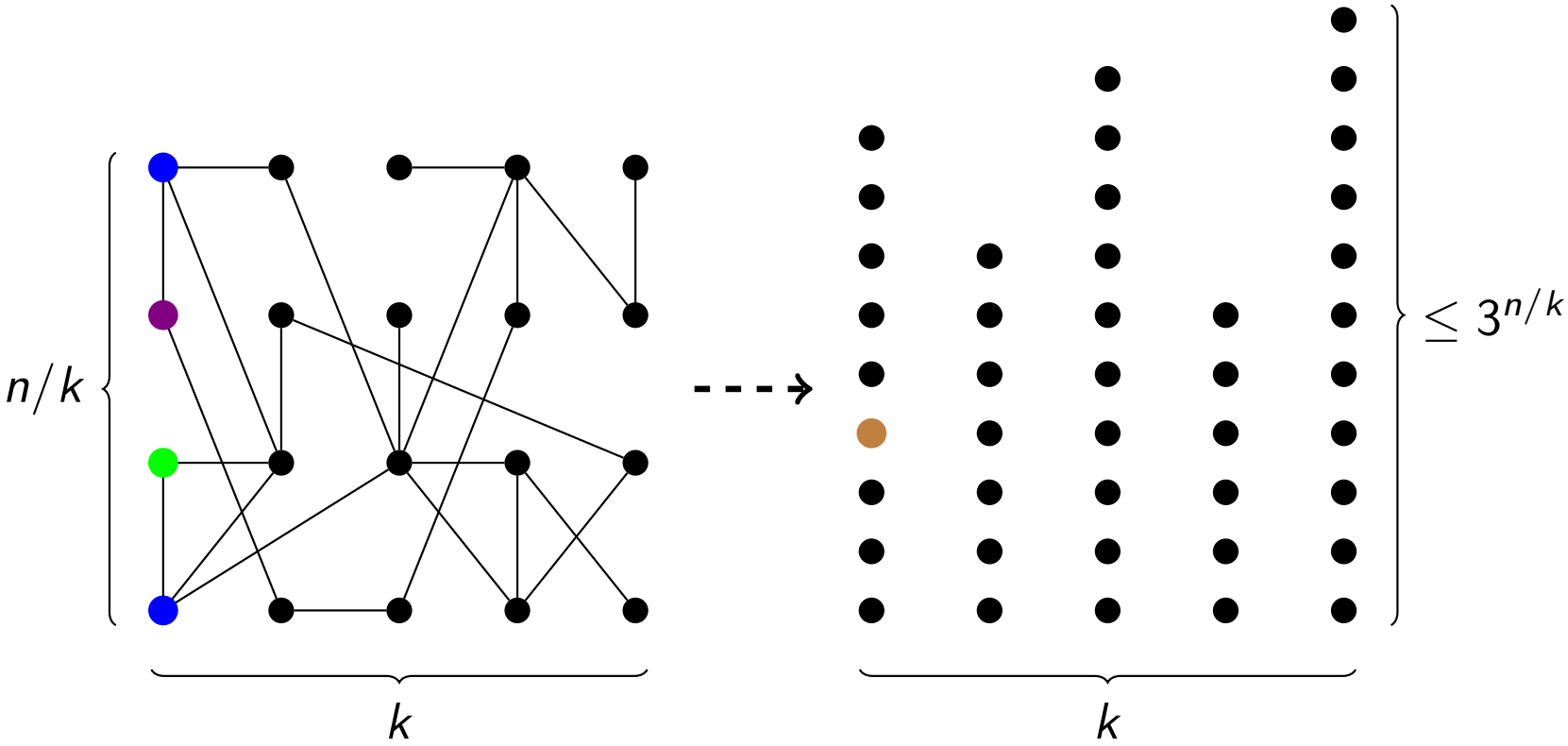

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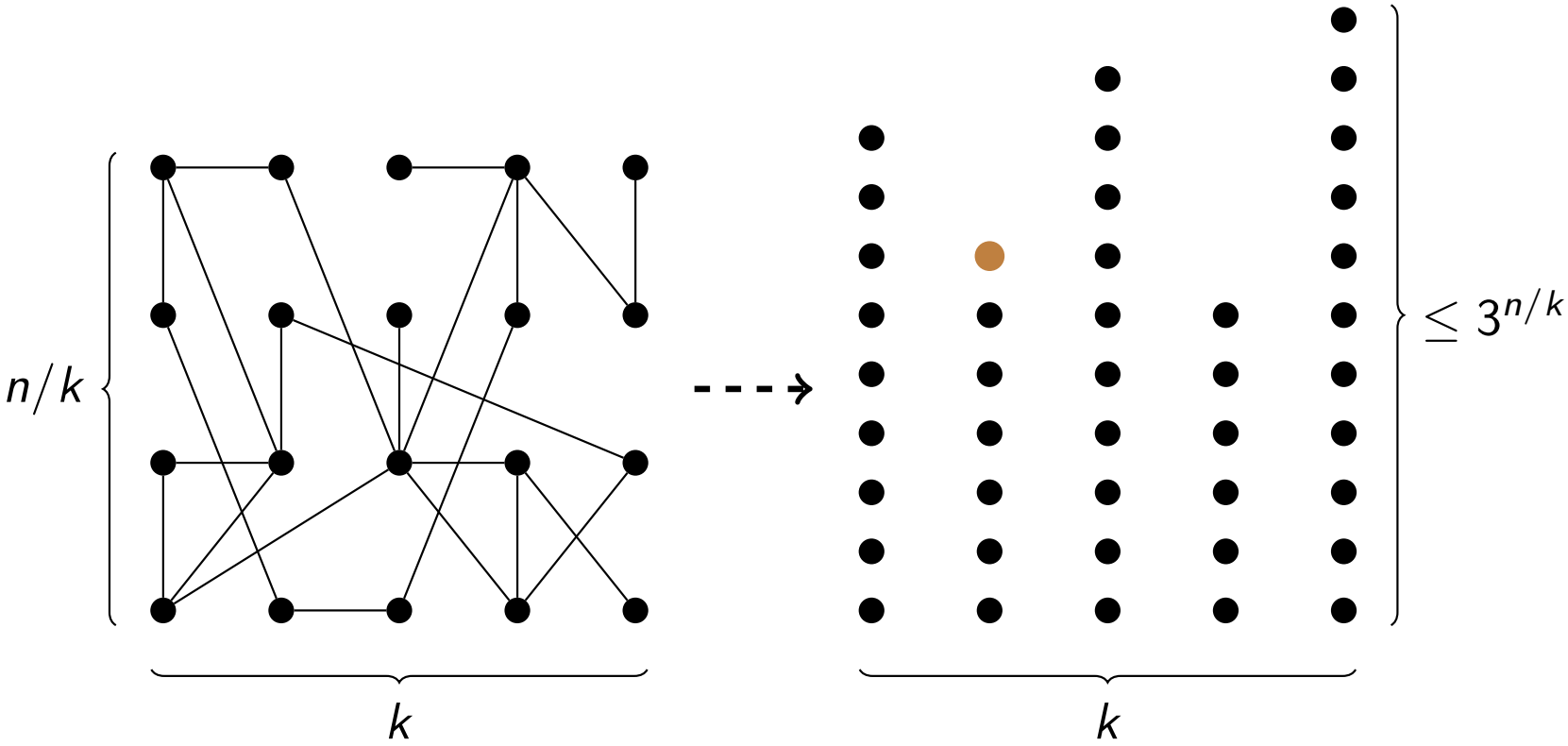

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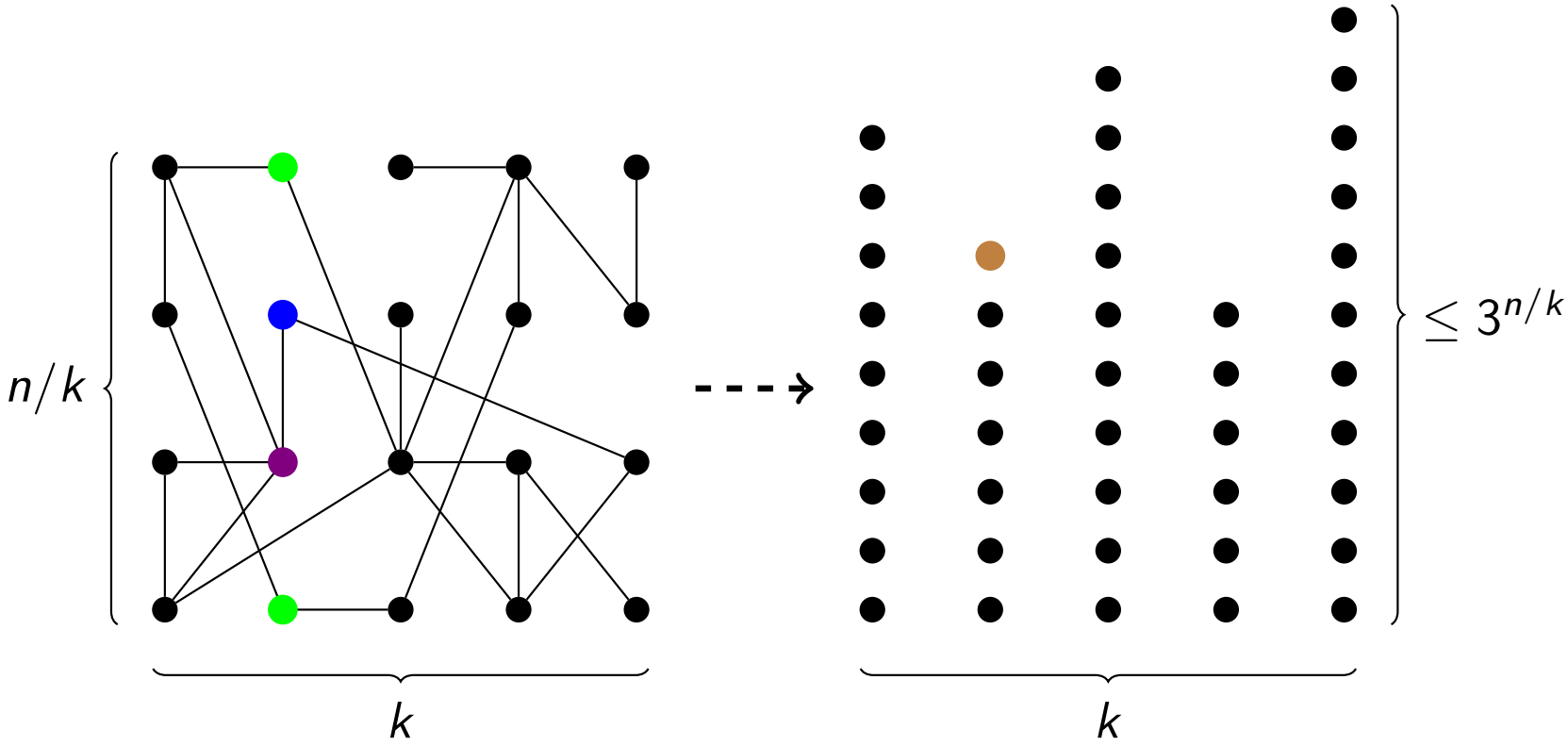

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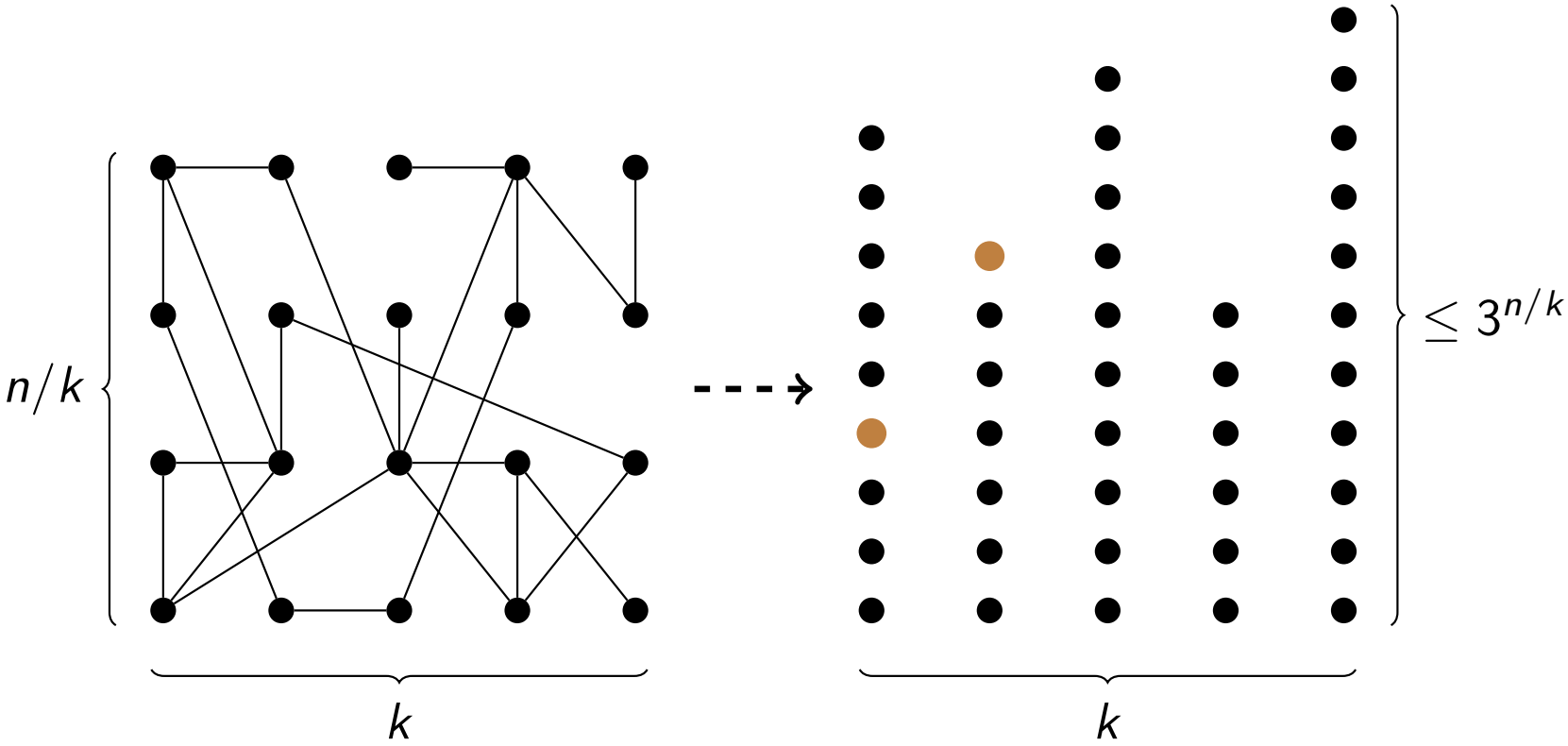

Connect two colorings if they are consistent.

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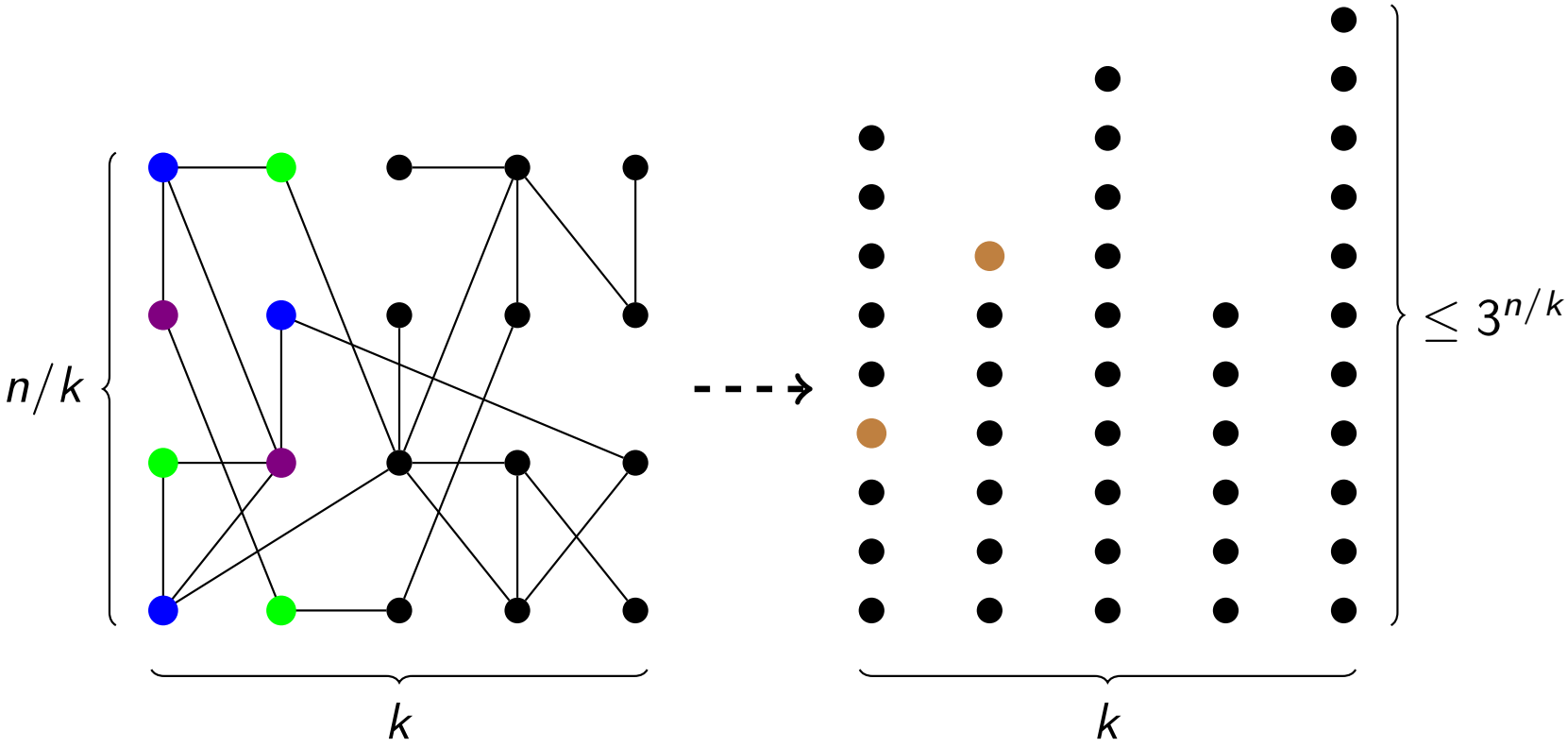

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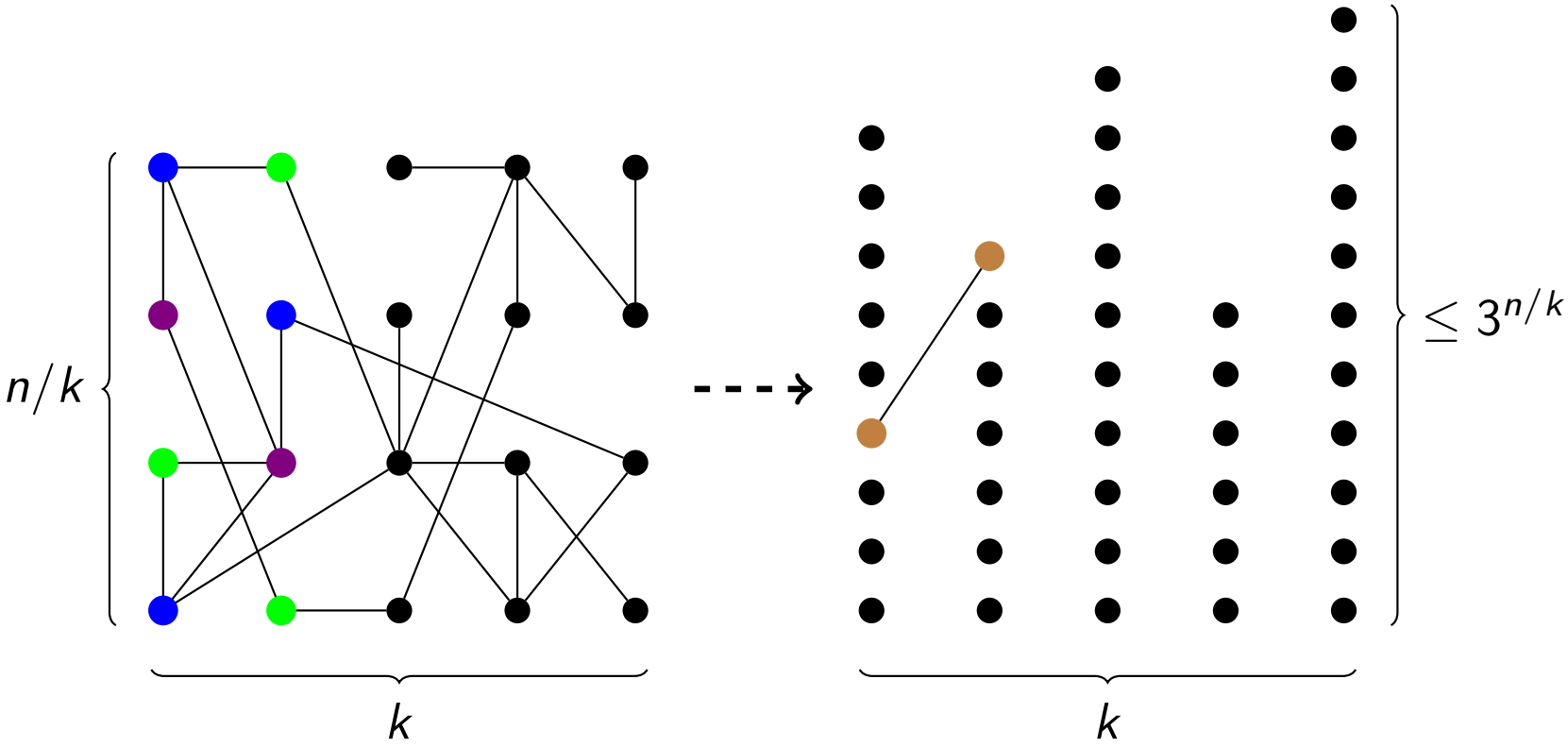

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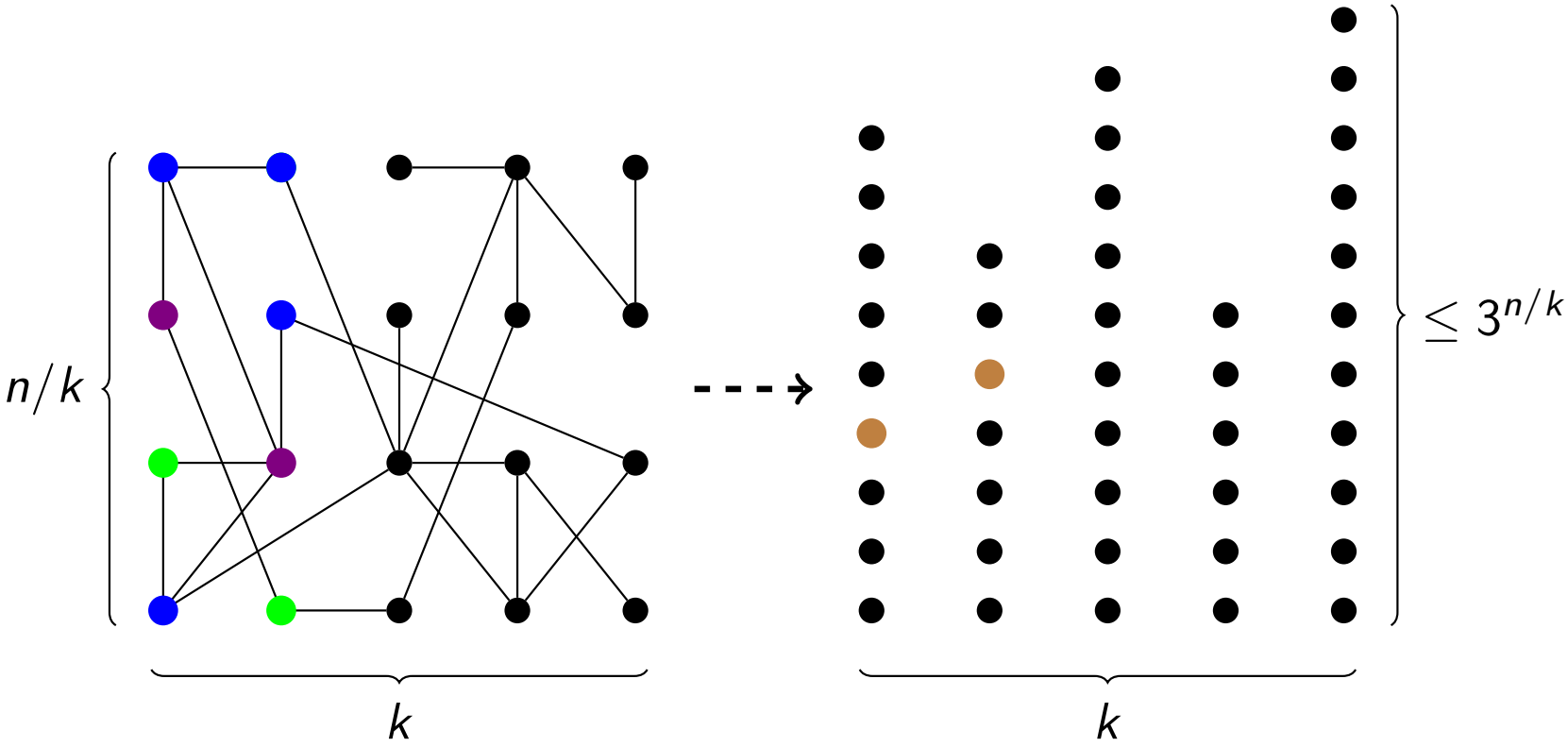

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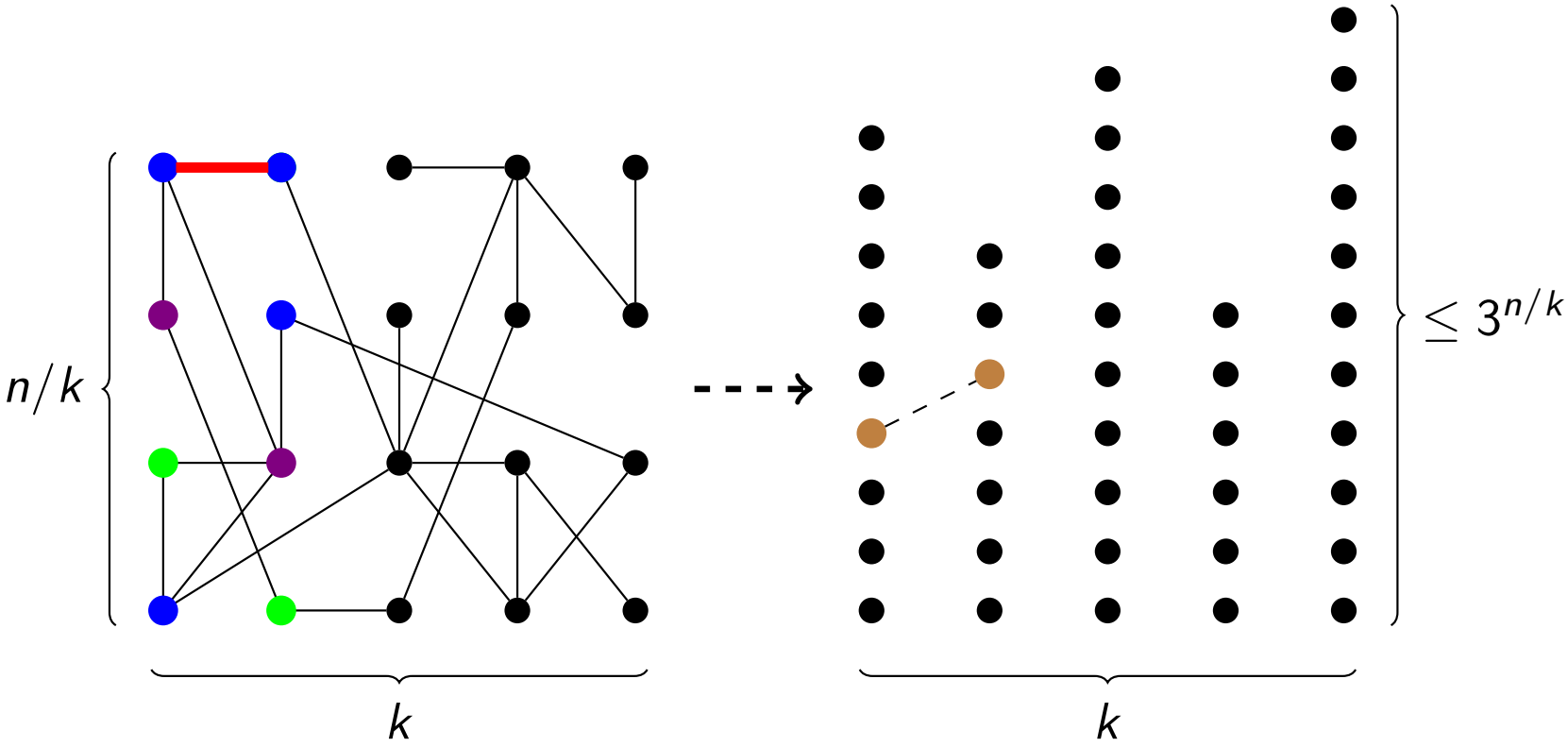

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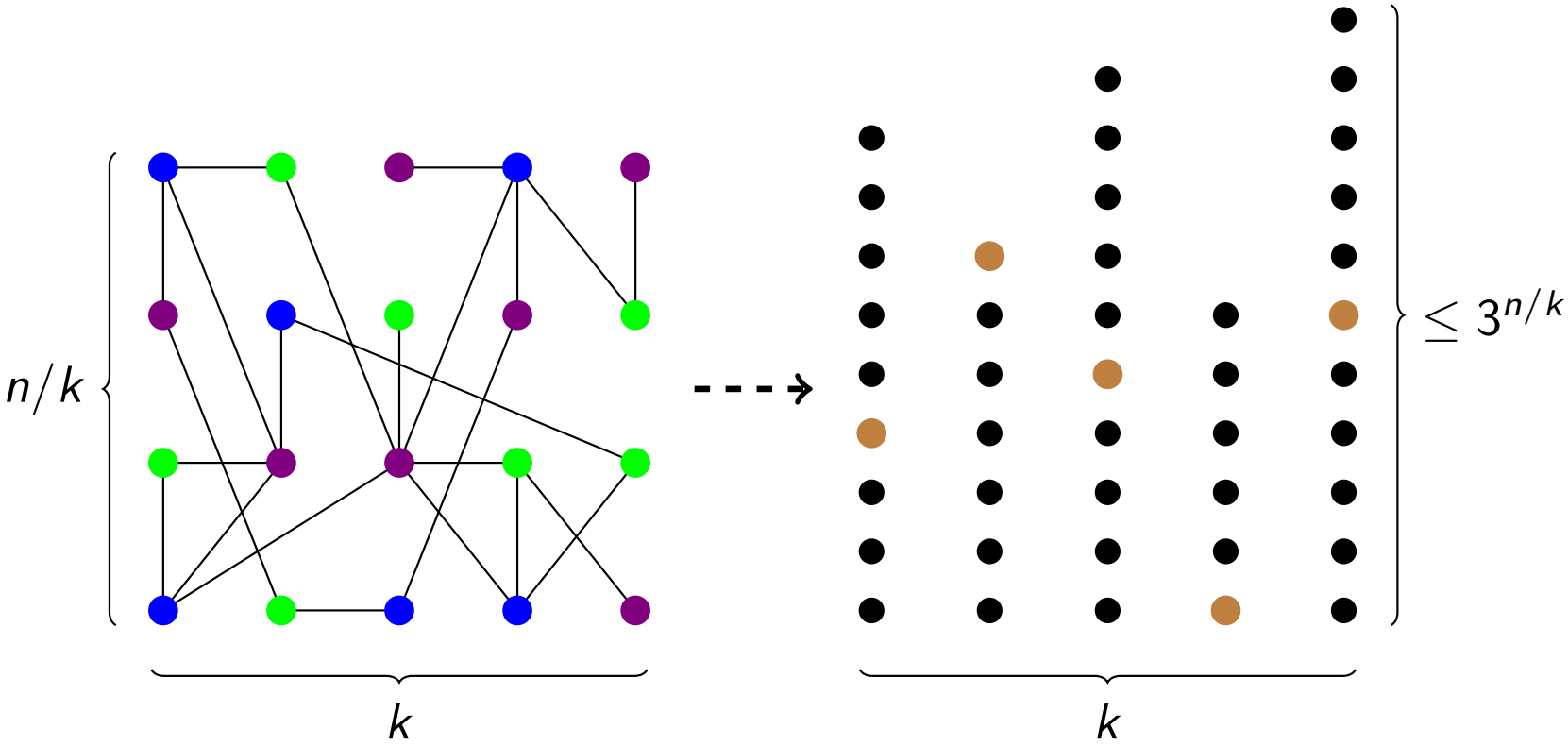

A 3-coloring on the left corresponds to a k-clique on the right.

Left graph admits 3-coloring iff right graph contains k-clique.

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- **Intuition**: We embed the 3^n solution space of 3-Coloring into the N^k solution space of Clique.

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- Under ETH, there is no algorithm with running time $f(k) \cdot n^{o(\sqrt{k})}$.

Corollaries

- By reductions from Clique, one can prove lower bounds on the running times of XP algorithms.
- **Parameter blow-up**: parameter k in a Clique instance is transformed into parameter $k' \leqslant g(k)$ for the output instance.
  - Linear blow-up $\rightsquigarrow$ $f(k') \cdot n^{o(k')}$ lower bound
  - Quadratic blow-up $\rightsquigarrow$ $f(k') \cdot n^{o(\sqrt{k'})}$ lower bound

Planar Scattered Set

I: An edge-weighted planar graph G, $d \in \mathbb{R}^+$, $k \in \mathbb{N}$
Q: Are there are k vertices pairwise at distance $\geqslant d$ from each other?

- There is an algorithm with running time $n^{\mathcal{O}(\sqrt{k})}$.
- Under ETH, there is no algorithm with running time $f(k) \cdot n^{o(\sqrt{k})}$.
  - The reduction transforms an instance of Clique with parameter k into an instance of Planar Scattered Set with parameter $\mathcal{O}(k^2)$.

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Lower bounds for parameterized problems under ETH

Suppose L admits a polynomial-time reduction from $3\mathrm{SAT}$ with output parameter $k \leqslant f(n+m)$.

Then L does not admit an $\mathcal{O}^\star(2^{o(f^{-1}(k))})$ time algorithm unless ETH fails.

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- We focus on (b), since this is the most typical behavior in parameterized algorithms.

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Q: Is there a k-clique in H that contains exactly one vertex from each row?

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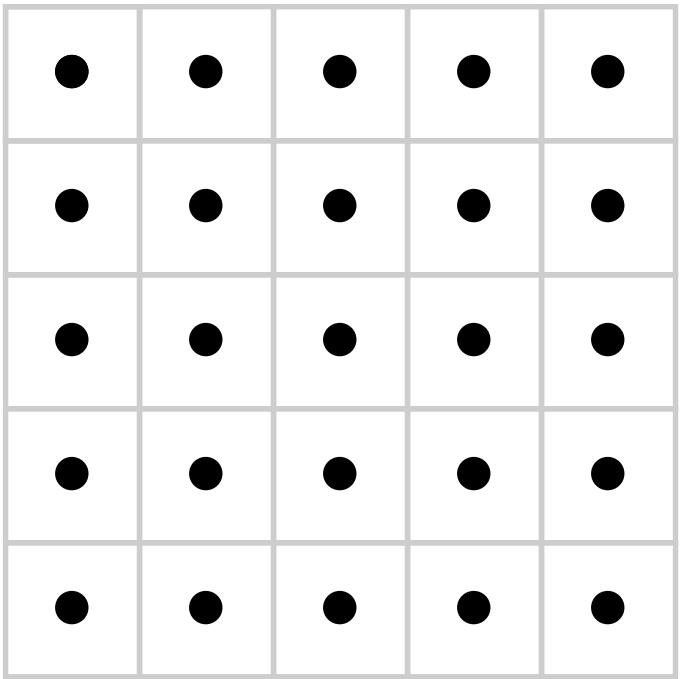

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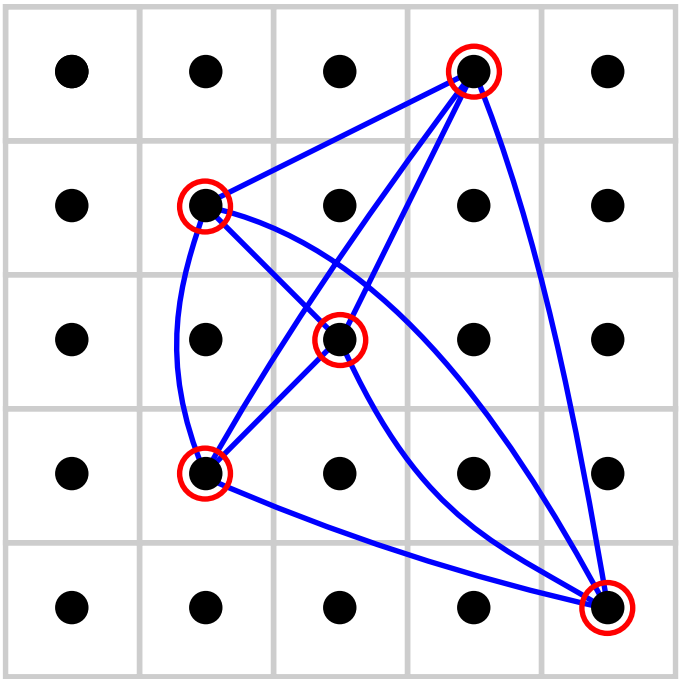

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Corollary

Unless ETH fails, there is no algorithm for $k \times k$-Clique with running time $\mathcal{O}^\star(2^{o(k \log k)})$.

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- **Further**: treewidth dynamic programming, e.g. Cycle Packing,

Another example: Edge Clique Cover

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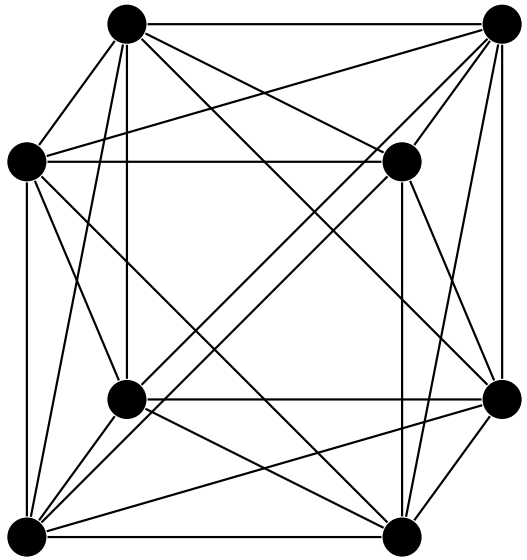

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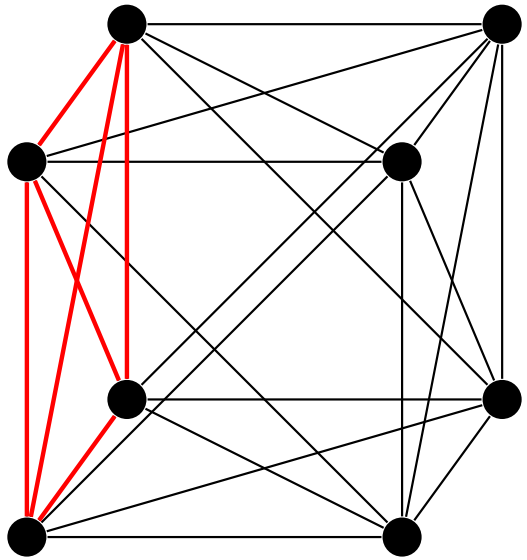

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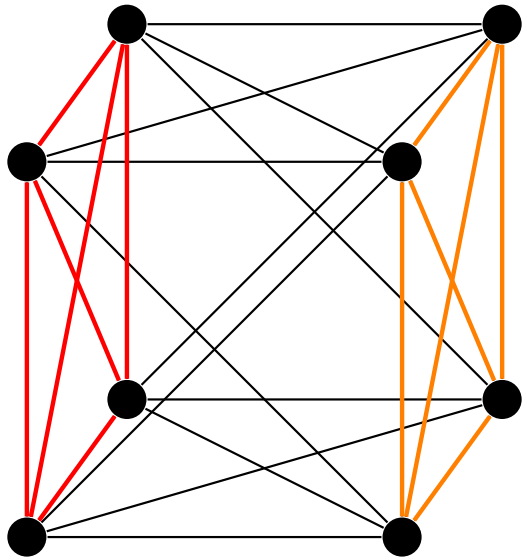

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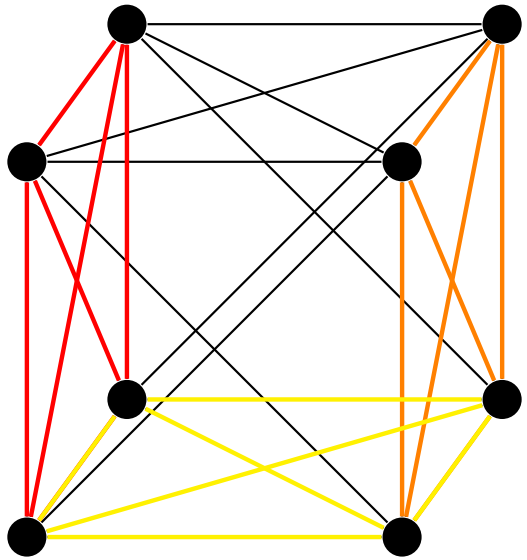

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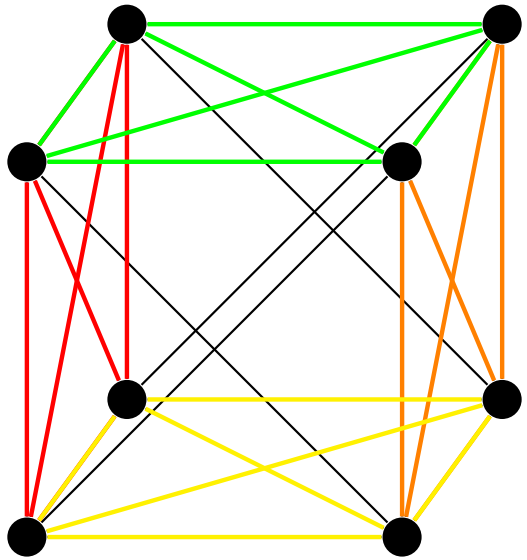

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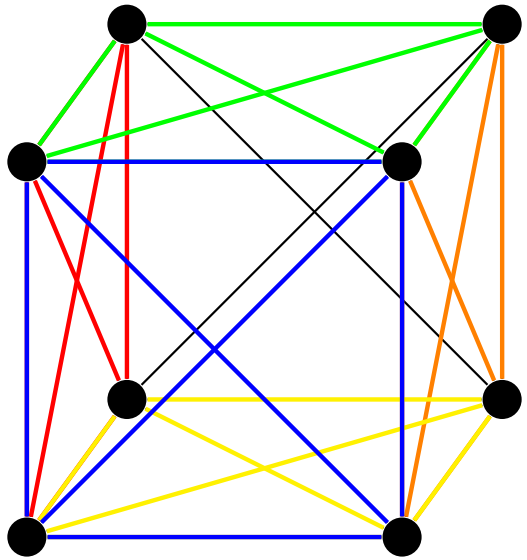

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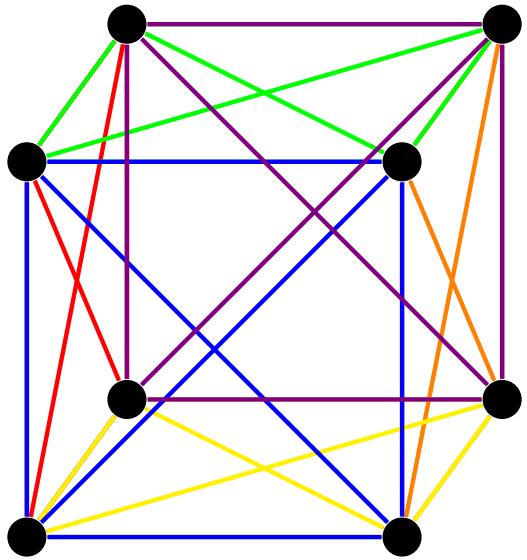

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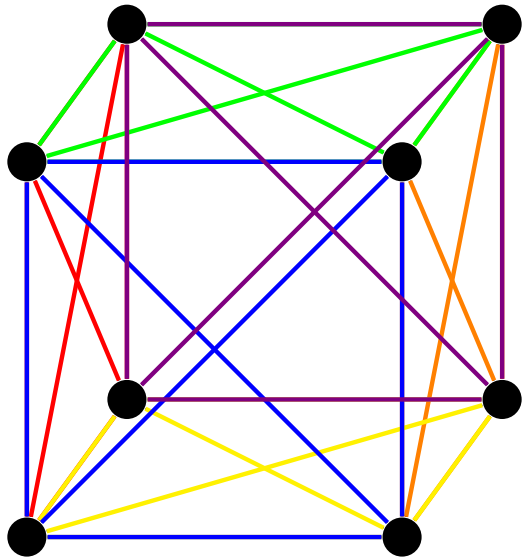

6 cliques: optimal?

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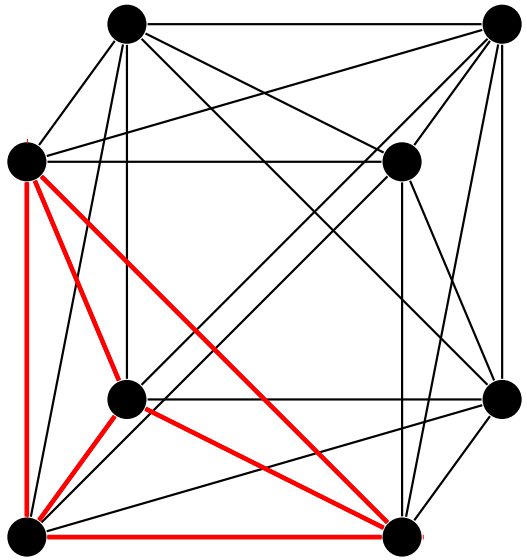

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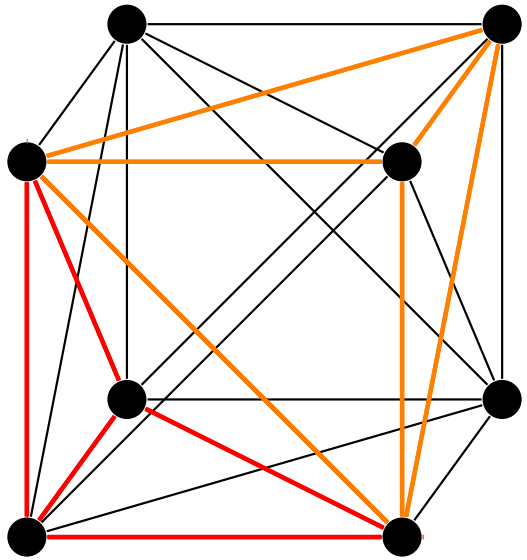

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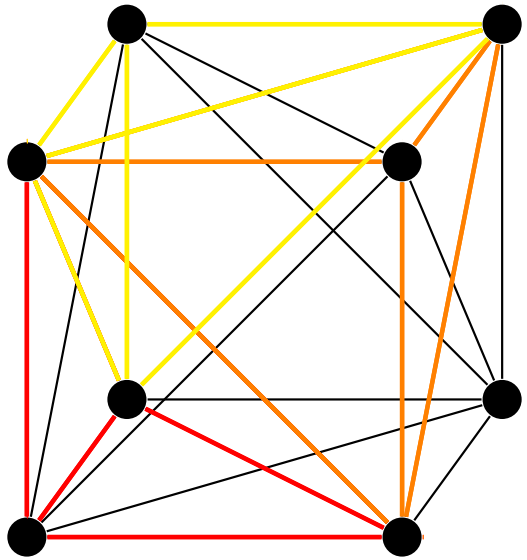

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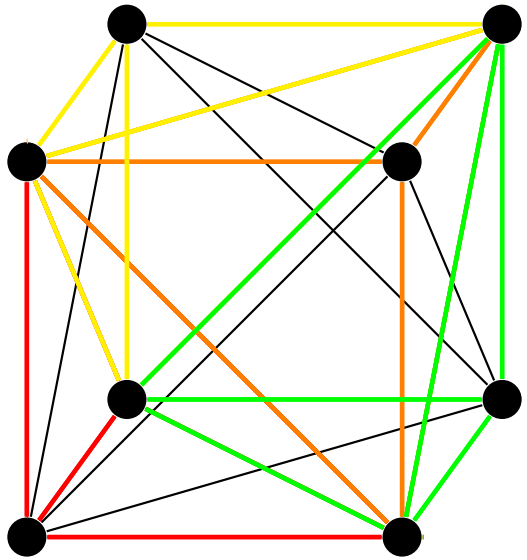

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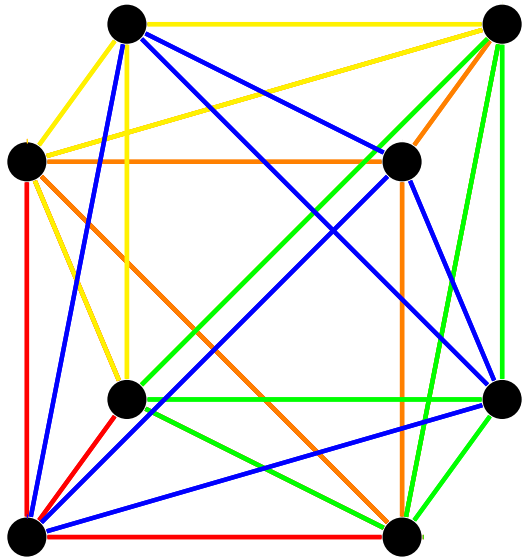

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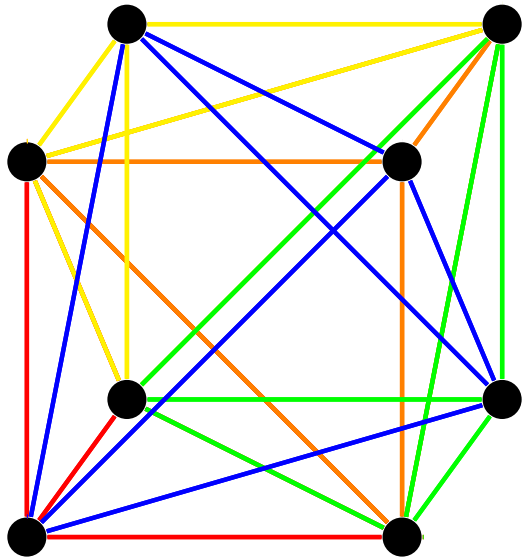

Non-trivial solution with 5 cliques

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- **Key idea**: If H is a clique of size $2n$ with a matching removed, then the optimum value for H is $\Theta(\log n)$.

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- More during Dániel's talk on Tueasday.

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- **Thank you for your attention!**