Lower bounds for polynomial kernelization

Part 1

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Recent Advances in Parameterized Complexity

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Kernelization — recap

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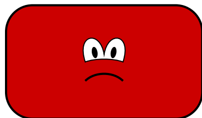

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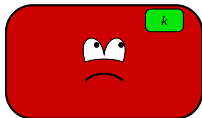

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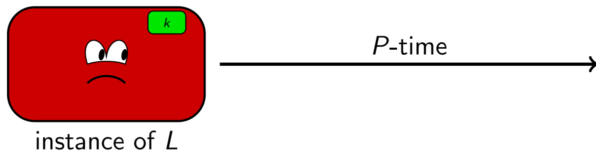

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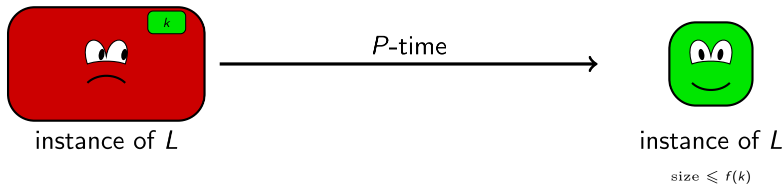

Background in complexity theory

Unparameterized problems

$\Leftrightarrow$

Languages over Σ, for a finite alphabet Σ

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Subsets of $\Sigma^\star$

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- **Unparameterized variant**: k is appended to x in unary.
- **Kernelization algorithm** takes on input an instance (x, k), and outputs an instance (x', k') such that

$$(x, k) \in L \Leftrightarrow (x', k') \in L \qquad \text{and} \qquad |x'| + k' \leqslant f(k)$$

for some computable function f.

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- Question of existence of **any kernel** is equivalent to being FPT.
- We are interested in **polynomial kernels**, where f is a polynomial.
- Before 2008, no tool to classify FPT problems wrt. whether they have polykernels or not.

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Intuition

The final number of bits is much less than the number input instances. Most of the instances have to be **discarded completely**.

Kernelization and Compression

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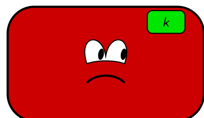

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P-time

instance of L
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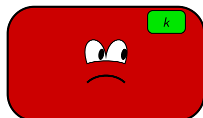

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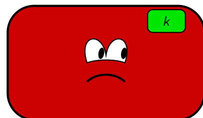

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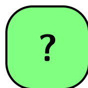

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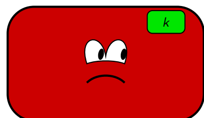

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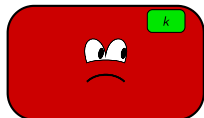

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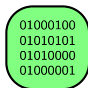

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- A polynomial compression can be turned into a polynomial kernelization provided that there is a **P**-reduction from R to L.
    - For instance, when $R \in \mathbf{NP}$ and L is **NP**-hard.
- **Note**: There are examples when a poly-compression is known but a poly-kernel is not known, because it is unclear whether R is in **NP**.

OR-distillation

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OR-distillation of L into R

Input: Words $x_1, x_2, \ldots, x_t$, each of length at most k.
Time: $\mathrm{poly}(t + \sum_{i=1}^{t} |x_i|)$.
Output: One word y such that

(a) $|y| = \mathrm{poly}(k)$, and
(b) $y \in R$ if and only if $x_i \in L$ for at least one i.

OR-distillation on picture

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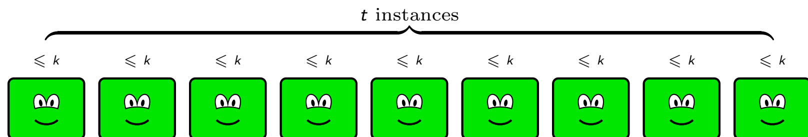

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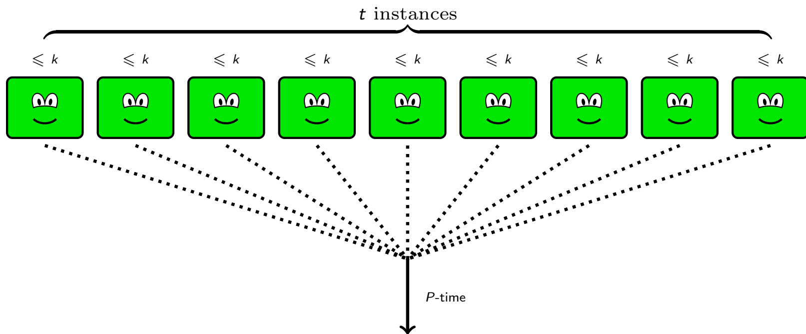

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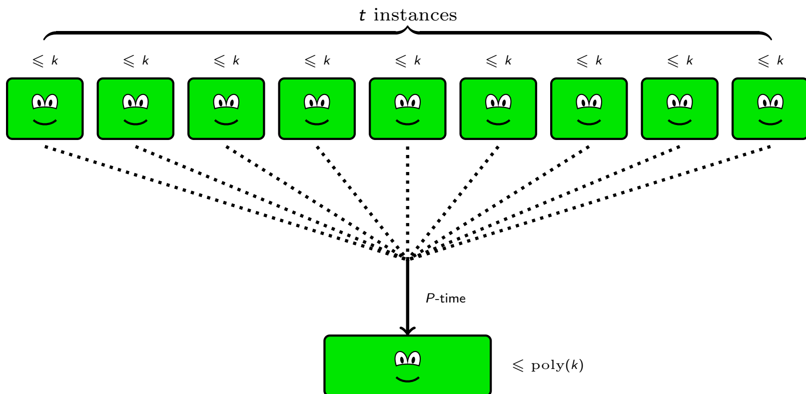

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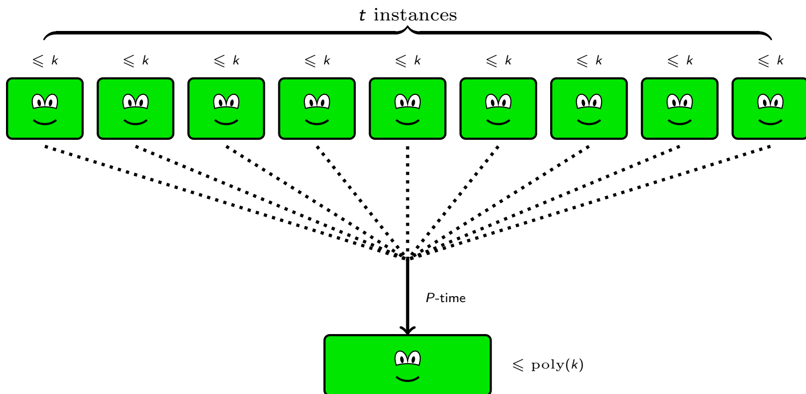

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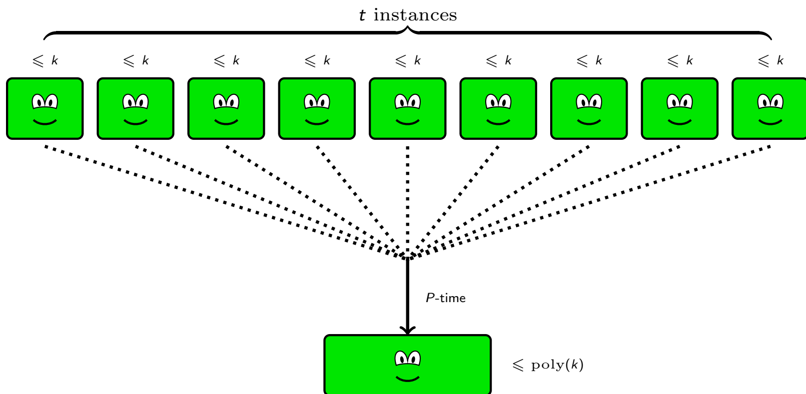

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Define $\mathrm{OR}\text{-}L = \{x_1\#x_2\#\ldots\#x_t \;:\; x_i \in L \text{ for at least one } i\}$.

OR-distillation $L \rightarrow R$ is a polynomial compression OR-$L/\max|x_i| \rightarrow R$

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OR-distillation theorem [Fortnow, Santhanam; 2008]

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- The proof is very short, but very tricky.

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- For details, look into the book.

OR-composition

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OR-composition algorithm for L

Input: Instances $(x_1, k), (x_2, k), \ldots, (x_t, k)$.
Time: $\mathrm{poly}(t + \sum_{i=1}^{t} |x_i| + k)$.
Output: One instance $(y, k^\star)$ such that

(a) $k^\star = \mathrm{poly}(k)$, and
(b) $(y, k^\star) \in L$ iff $(x_i, k) \in L$ for at least one i.

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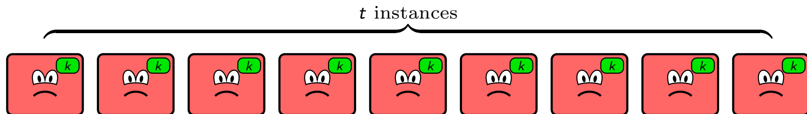

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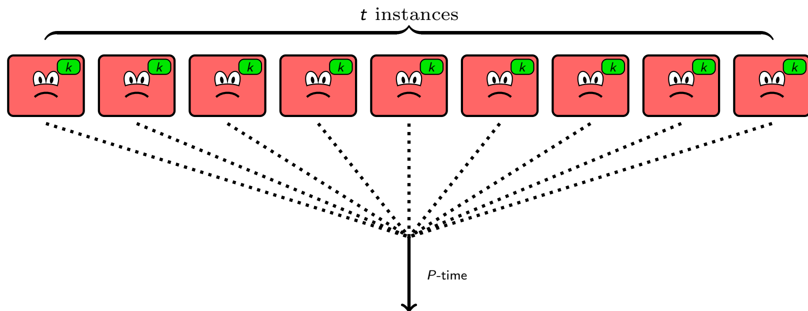

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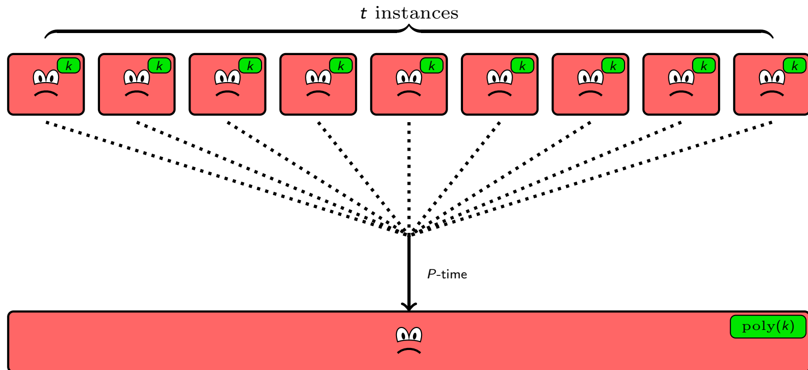

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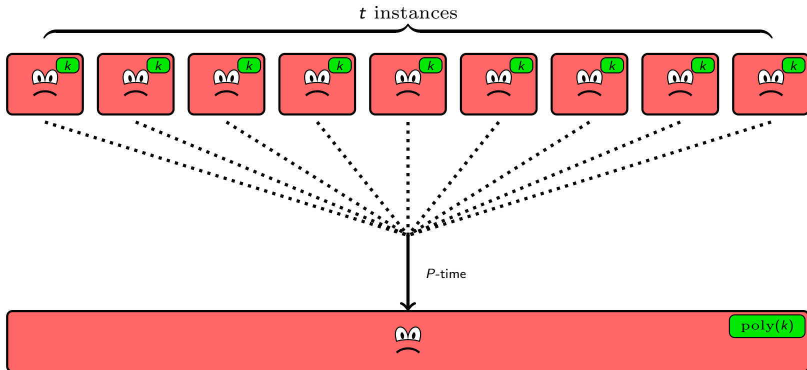


OR-composition theorem [BDFH; 2008]

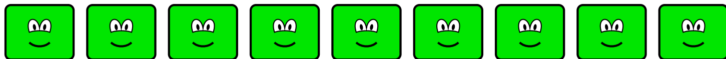
Suppose a parameterized problem L admits an OR-composition algorithm, and the unparameterized version of L is **NP**-hard.

Then L does not admit a polynomial kernel unless $\mathbf{NP} \subseteq \mathbf{coNP}/\mathrm{poly}$.

Proof

Proof

OR-SAT



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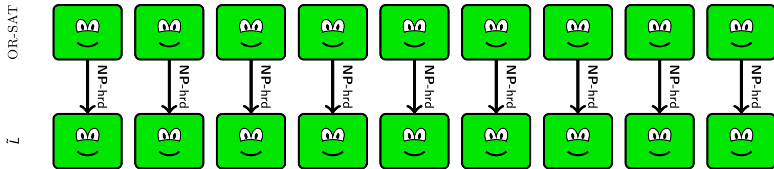

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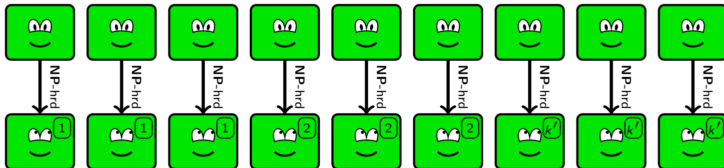

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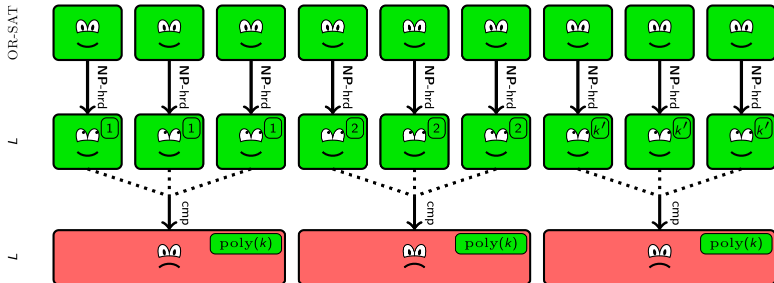

Proof

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NP-hrd

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1 1 1 2 2 2 k' k' k'

cmp

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poly(k) poly(k) poly(k)

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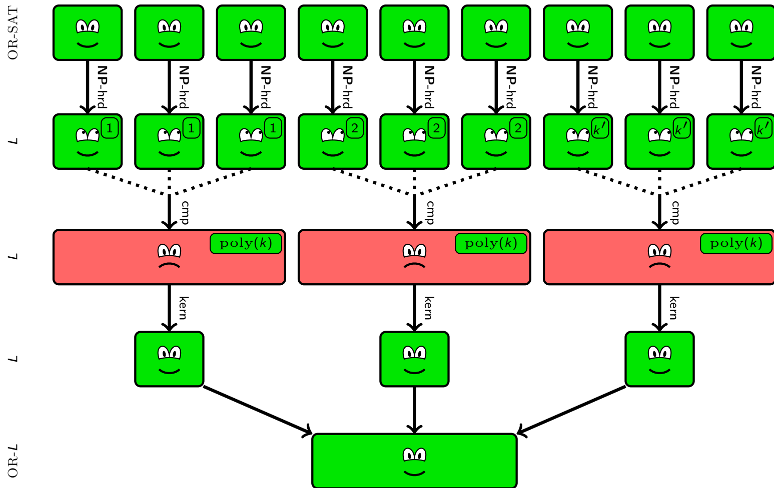

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- Same for k-Cycle and many other problems.
- Today, investigating the existence of a polynomial kernel is often a secondary goal after showing that a problem is FPT.

Adding features

- Does the proof actually exclude even polynomial compression into any R, not just kernelization?

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  - Ergo, the parameter of the composed instance may depend polynomially on **both** k and $\log t$.

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- Most of the works use a subset of mentioned features.
- **Later**: a new formalism **cross-composition** gathers all the features. (Bodlaender, Jansen, and Kratsch; 2011)

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Equivalence relation $\sim$ on $\Sigma^\star$ is a **polynomial equivalence relation** if:

- checking whether two words $x, y \in \Sigma^\star$ are $\sim$-equivalent can be done in $\mathrm{poly}(|x| + |y|)$ time; and
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  - or with respect to *(i)* the number of vertices, *(ii)* the number of edges, *(iii)* size of the maximum matching, *(iv)* budget.

Cross-composition

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An unparameterized problem Q **cross-composes** into a parameterized problem L, if there exists a polynomial equivalence relation $\sim$ and an algorithm that, given $\sim$-equivalent strings $x_1, x_2, \ldots, x_t$, in time $\mathrm{poly}\left(t + \sum_{i=1}^{t} |x_i|\right)$ produces one instance $(y, k^\star)$ such that

- $(y, k^\star) \in L$ iff $x_i \in Q$ for at least one $i = 1, 2, \ldots, t$,
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Cross-composition theorem

[Bodlaender, Jansen, Kratsch]

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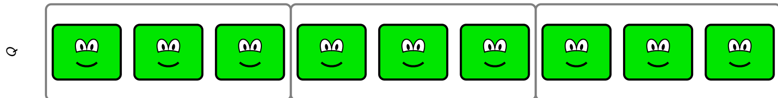

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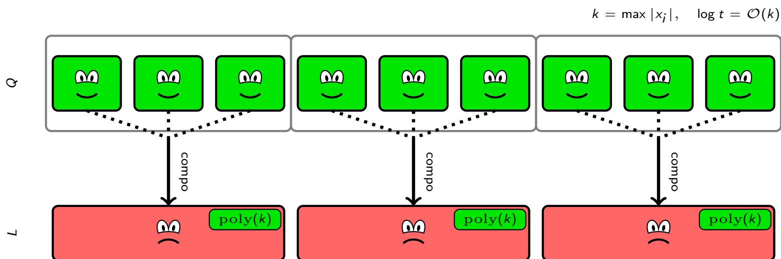

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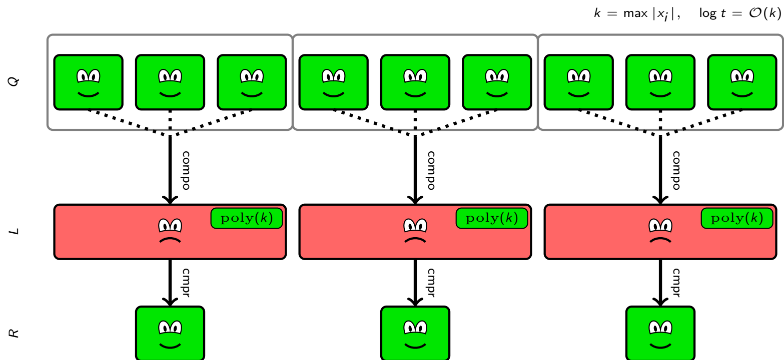

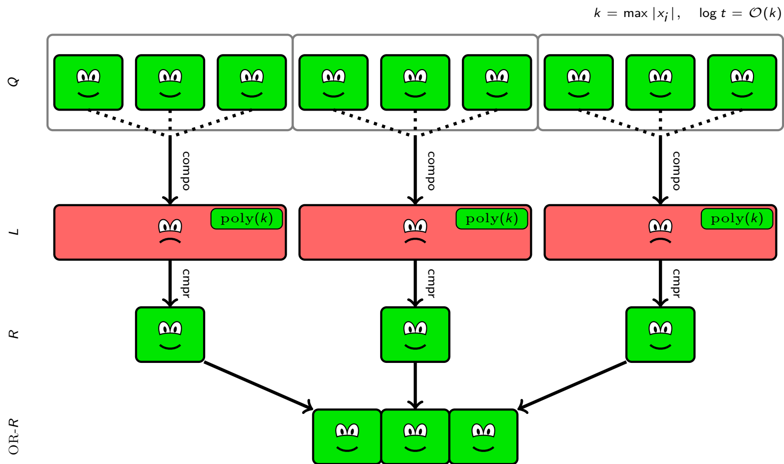

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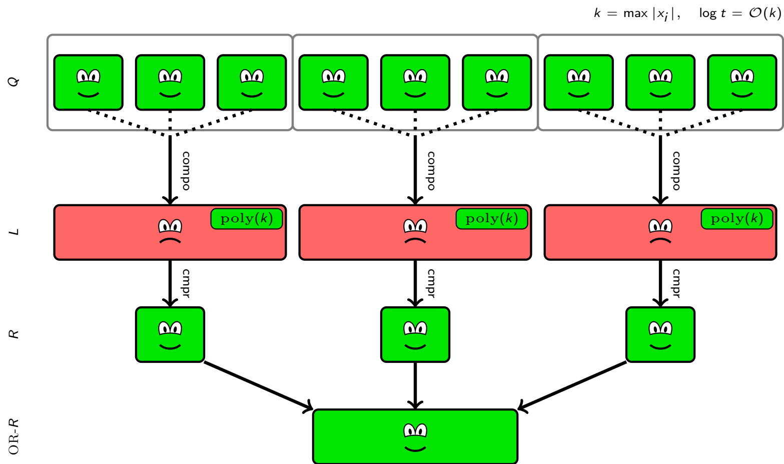

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- In fact, cross-composition is a good framework to express also all the previous results.
- **Plan for now**: show some non-trivial cross-composition to give an intuition about basic tricks.

Application: Set Splitting

Set Splitting

I: Universe U and family of subsets $\mathcal{F} \subseteq 2^U$
P: $|U|$
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- Assume that t is a power of 2 (by copying the instances).

Cross-composing into Set Splitting

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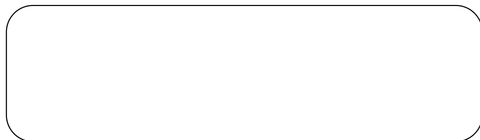

PLAYGROUND

joint universe U

Cross-composing into SET SPLITTING

Input: Instances $(U, \mathcal{F}^i)$

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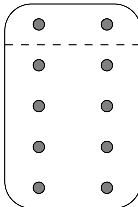
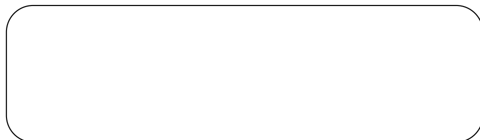
INSTANCE SELECTOR

$1 + \log t$ pairs of vertices





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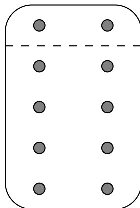

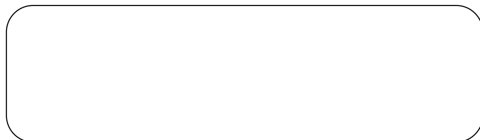

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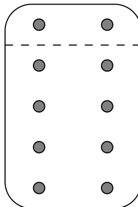

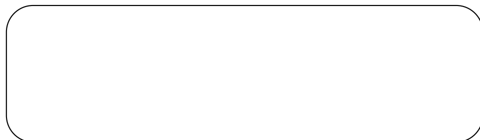

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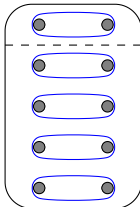

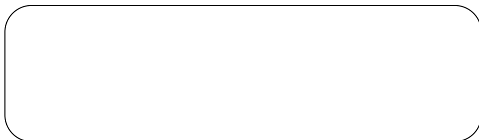

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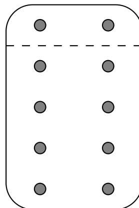

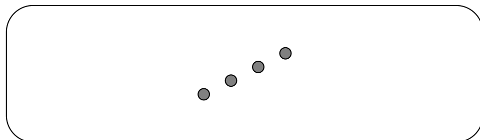

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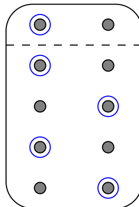

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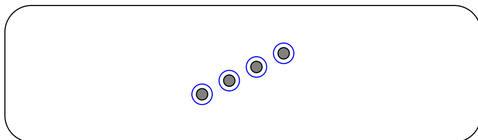

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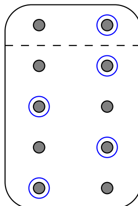

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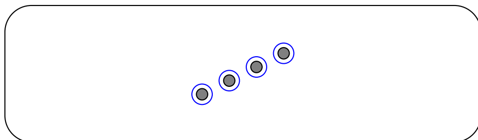

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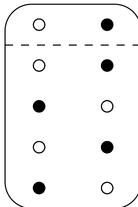
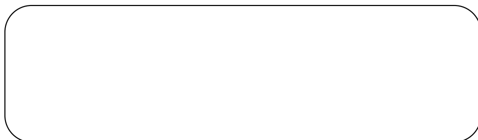


Take any solution $\mathcal{C}$



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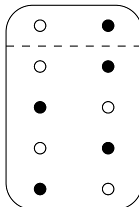

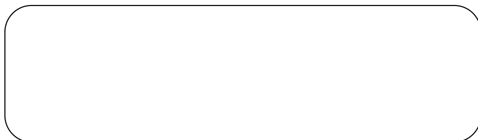

PLAYGROUND

joint universe U

Take any solution $\mathcal{C}$

There is exactly one index i with monochromatic parts from IS.

Cross-composing into Set Splitting

Input: Instances $(U, \mathcal{F}^i)$

Output: Instance $(U^*, \mathcal{F}^*)$

$|U^*| = |U| + 2\log t + 2$

$\mathcal{F}^*$ **consists of**:
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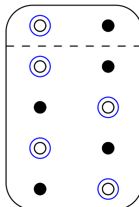

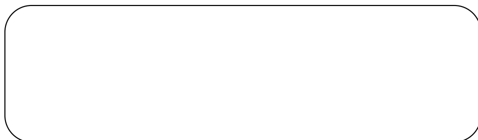

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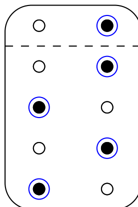

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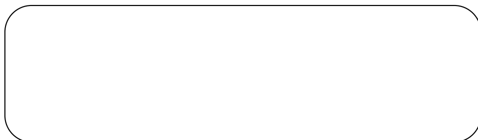

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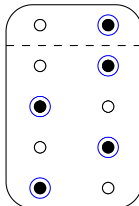

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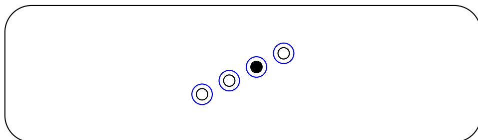

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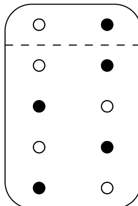

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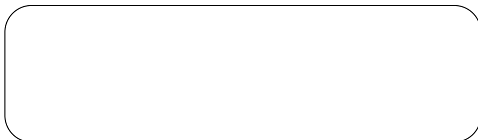
($\Leftarrow$): If $(U, \mathcal{F}^i)$ is solvable, we set IS accordingly, and solve this instance in PL. Remaining sets are split for free.



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  - Choice of the index makes the instance active, while the other instances are "switched off".

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- **Thank you for your attention!**

Tikz faces based on a code by Raoul Kessels, http://www.texample.net/tikz/examples/emoticons/,

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