Lower bounds for polynomial kernelization

Part 2

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Recent Advances in Parameterized Complexity

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PPTs

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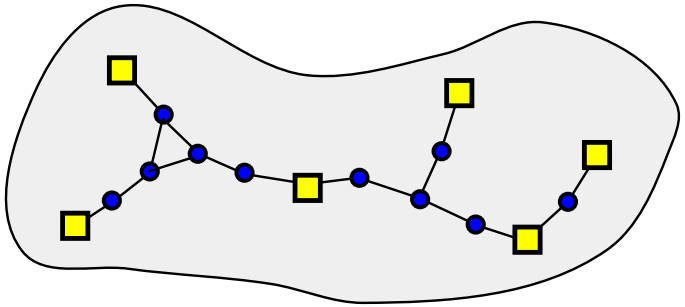
- **Proof**:
  Compose the PPT with the assumed compression for Q.

Application: Steiner Tree

Steiner Tree

I: Graph G with terminals $T \subseteq V(G)$, $k \in \mathbb{N}$
P: $k + |T|$
Q: Is there a set $X \subseteq V(G) \setminus T$, such that $|X| \leqslant k$ and $G[T \cup X]$ is connected?



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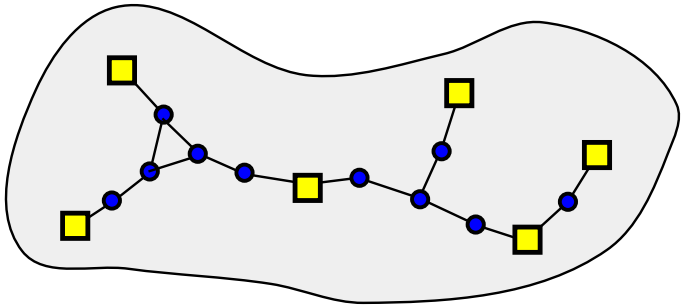
- We show that Steiner Tree has no polykernel (unless...) using a PPT from a auxiliary problem.



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- **Idea**: Move the weight of the proof to the transformation and the actual definition of P.
- **High level**: Extract the essence of the original problem into the auxiliary problem.

Colorful Graph Motif

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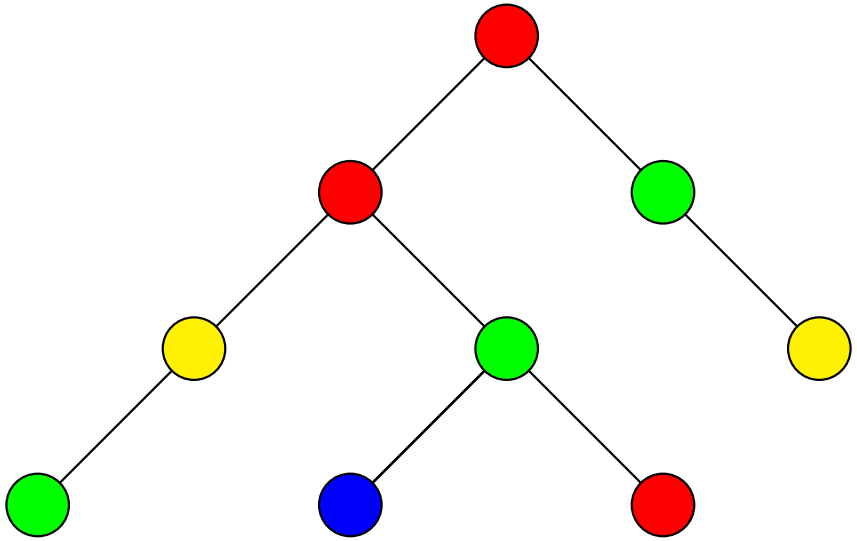

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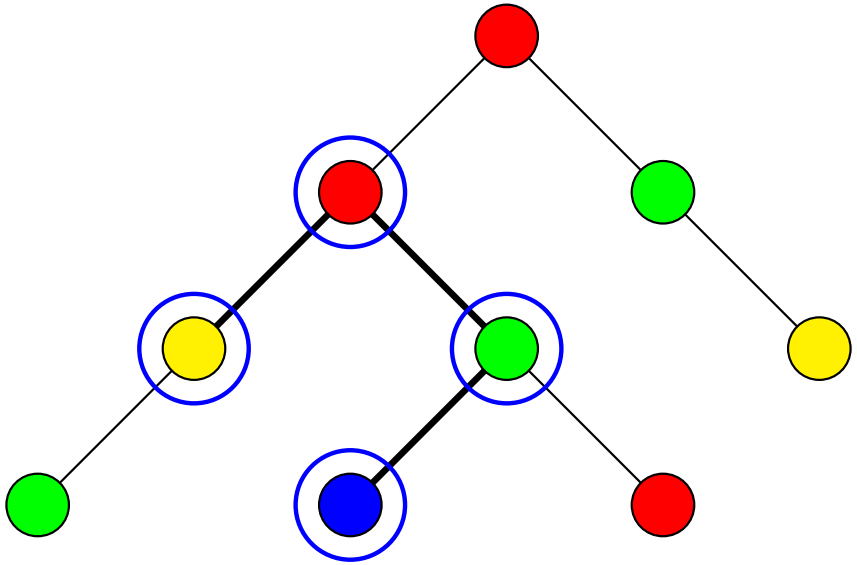

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- **Now**: PPT from CGM to ST.

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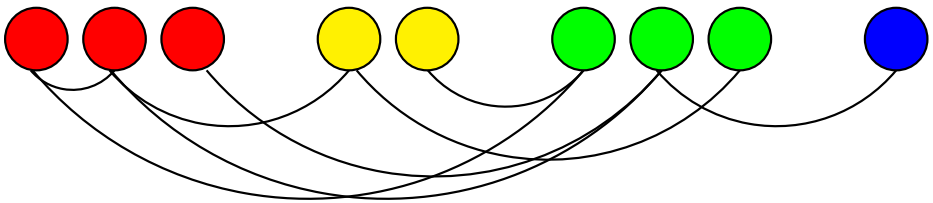

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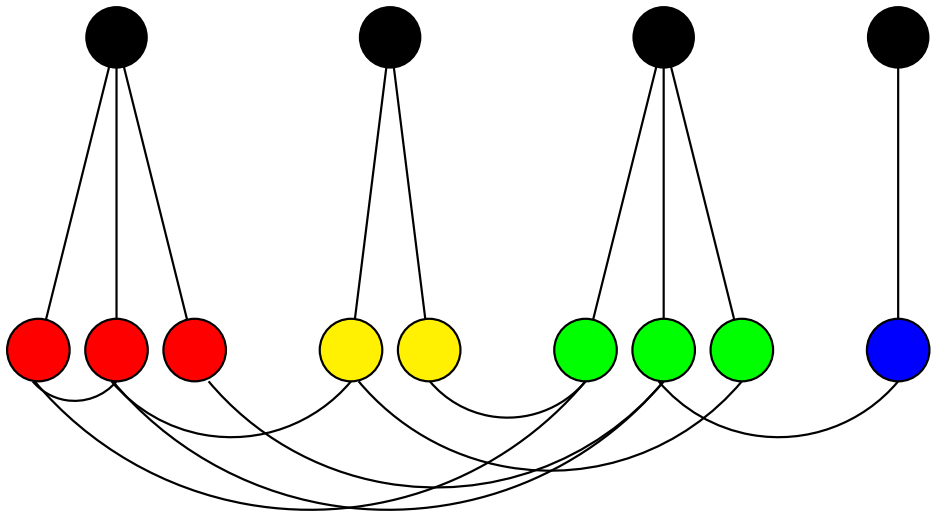

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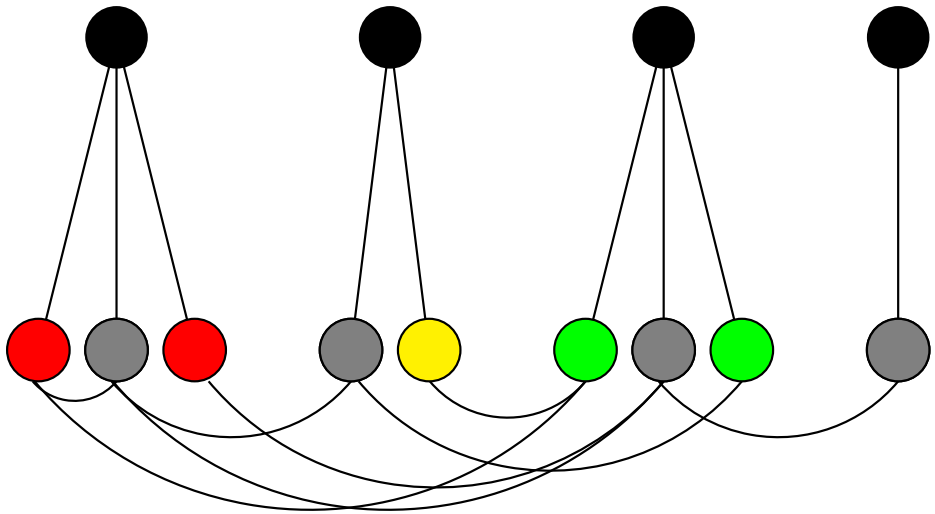

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- Hence Steiner Tree par. by $k + |T|$ does not admit a polynomial kernel, unless **NP** $\subseteq$ **coNP**/poly.
- **Note**: Composition for CGM is far simpler than trying to do this directly for Steiner Tree.

Another example: Disjoint Paths

Disjoint Paths

I: Graph G with k terminal pairs $(s_1, t_1), \ldots, (s_k, t_k)$

P: k

Q: Is there a **linkage** consisting of vertex-disjoint paths $P_1, \ldots, P_k$, where P_i connects s_i with t_i?

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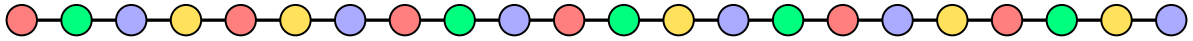

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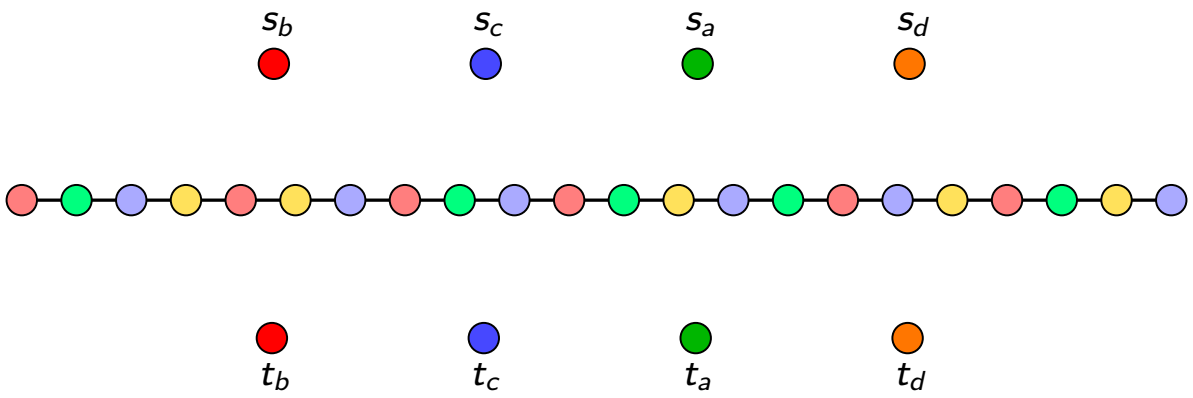

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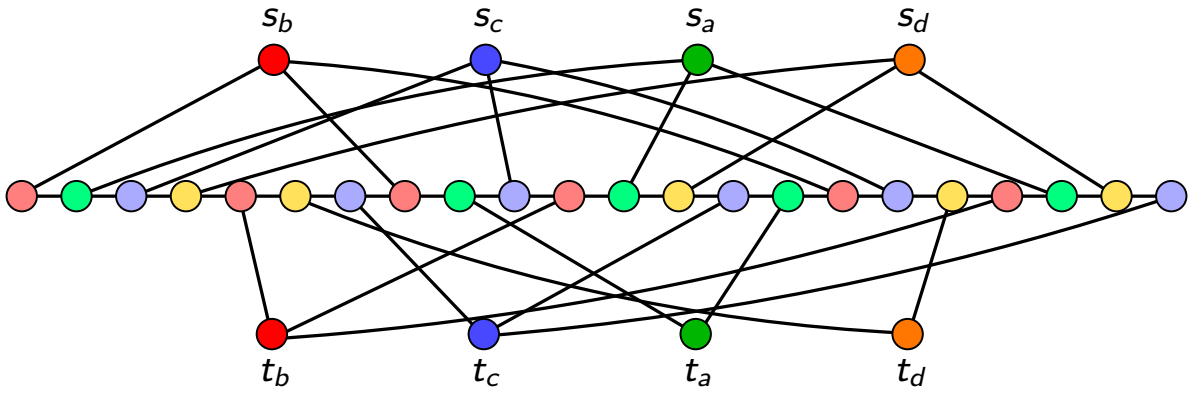

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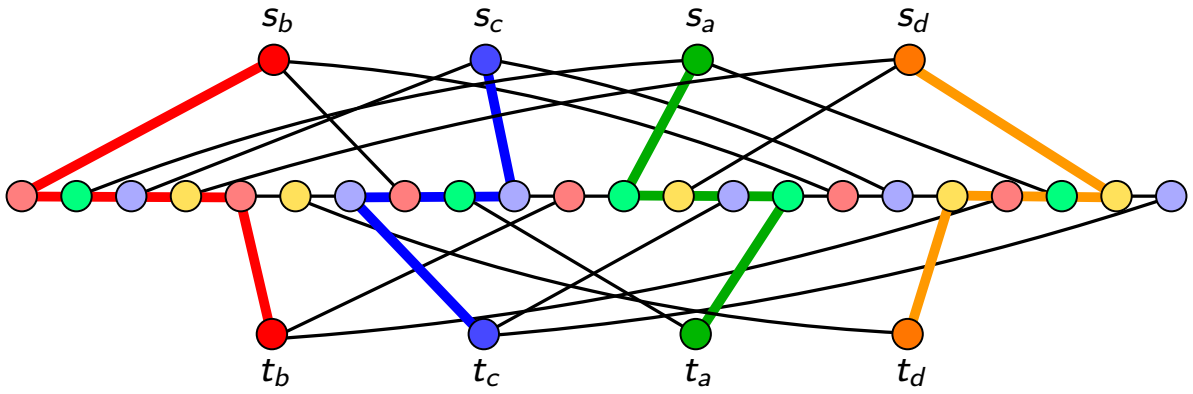

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  - The conjecture was proved by Drucker in 2012.

AND-compositions

- In the compositionality framework, we used the OR function to compose instances.
- What about replacing it with, say, AND?
- **AND-distillation**, **AND-(cross)-composition**:
  Same as before, but with AND instead of OR.
  - Example of problem admitting an AND-composition:
    Is the treewidth of a given graph at most k?
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- **Corollary**: The whole framework works for AND instead of OR.

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- First observed by Dell and van Melkebeek in 2010, then developed by other authors as well.

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- Let's look again at the proof of the cross-composition theorem.

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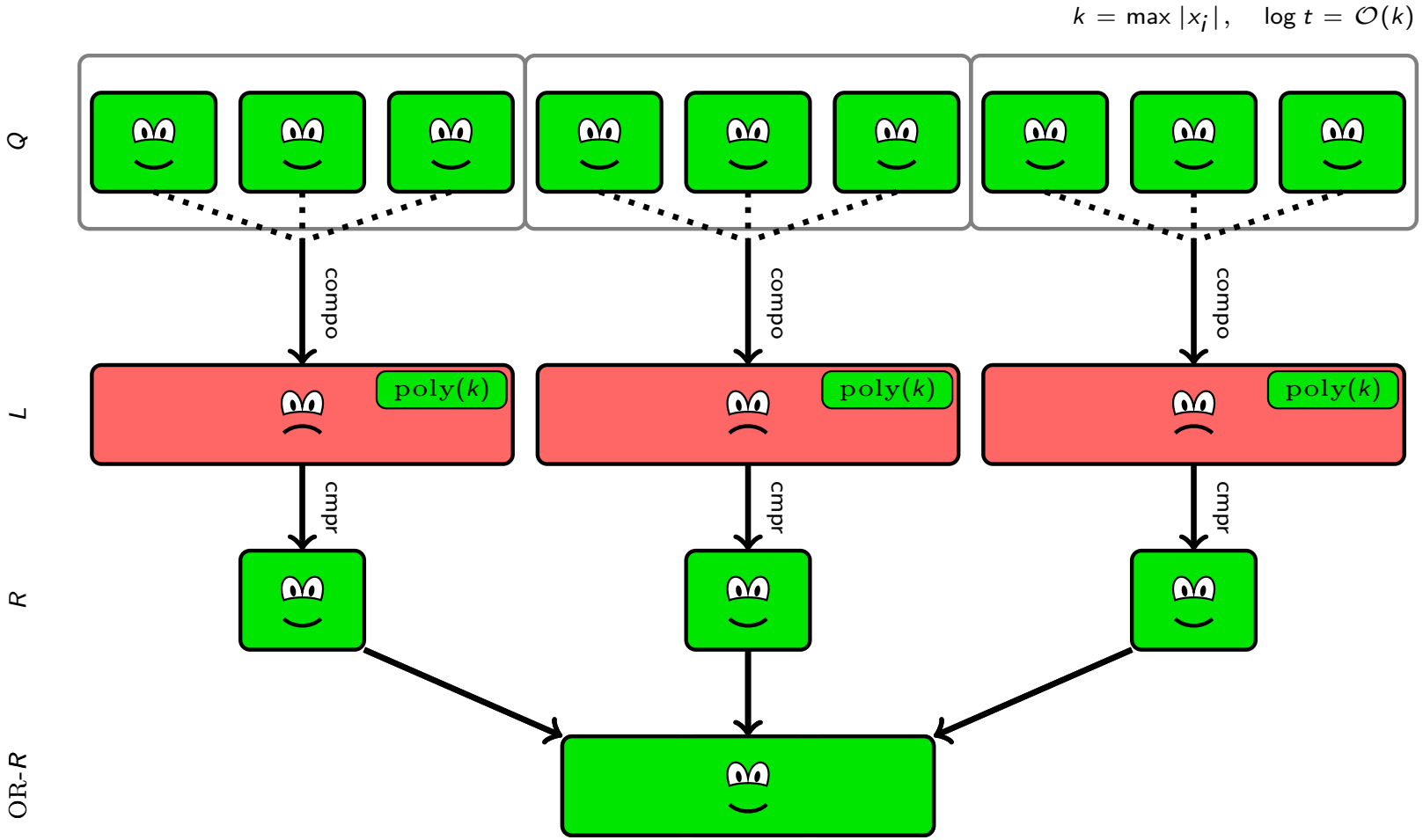

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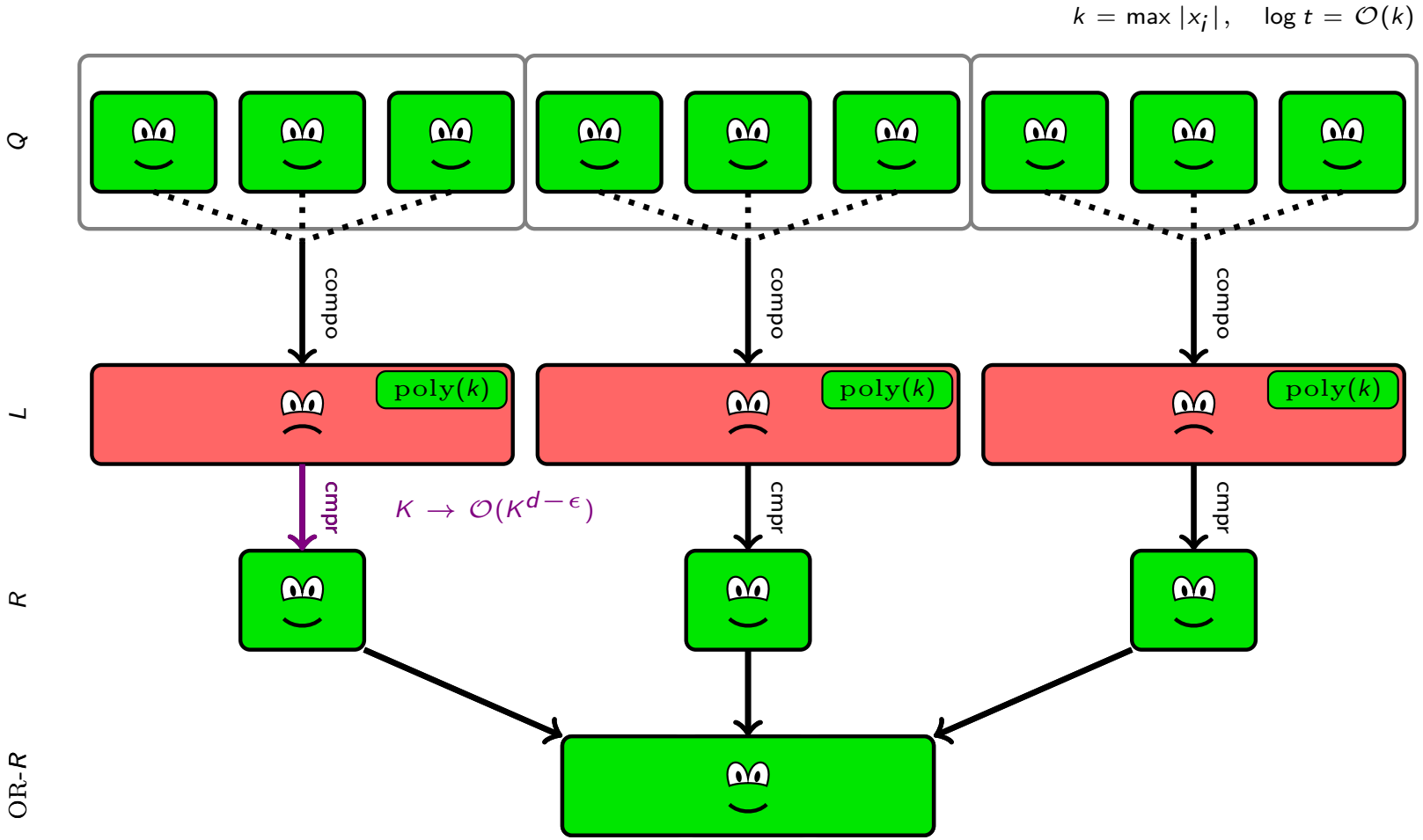

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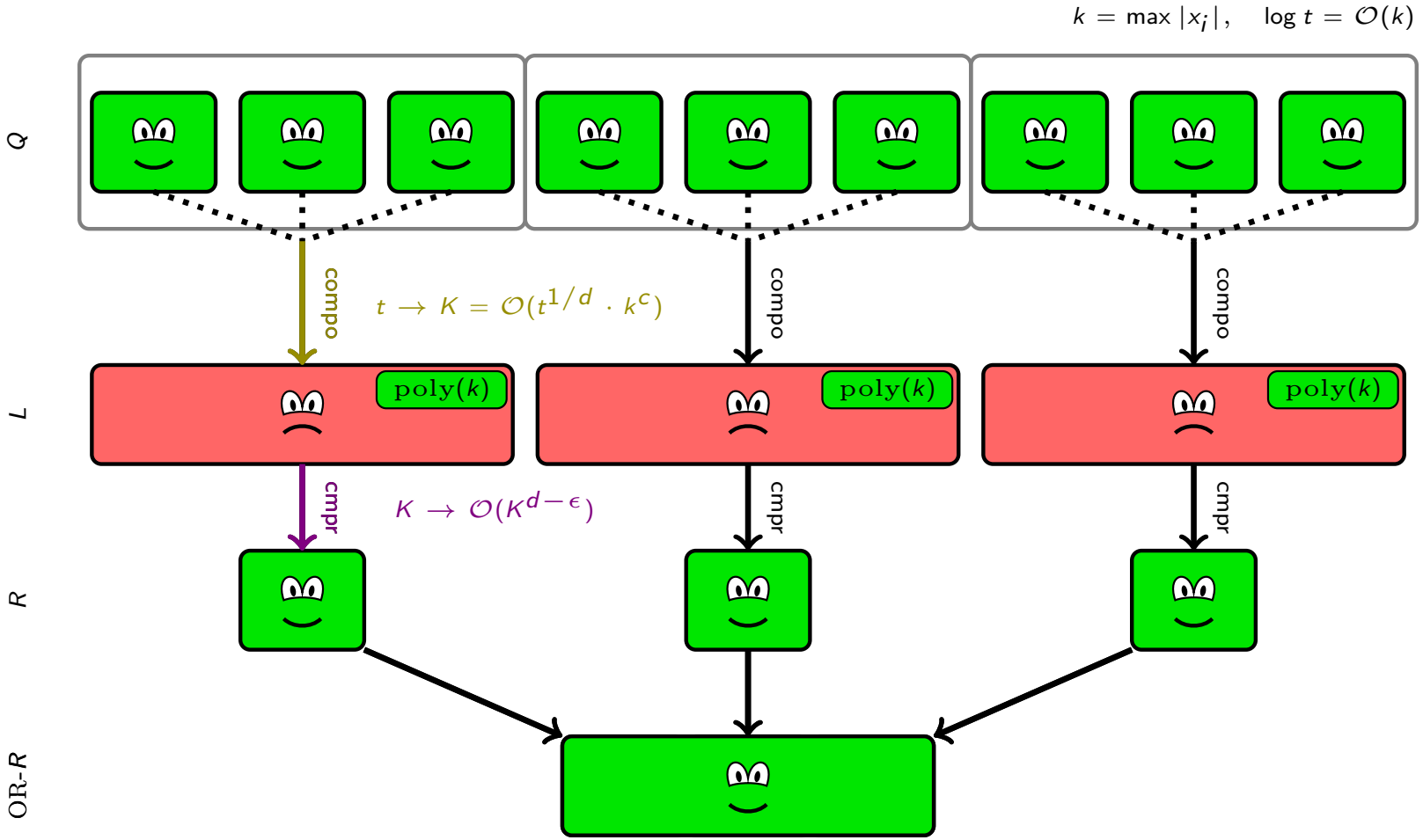

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An unparameterized problem Q **weakly cross-composes** into a parameterized problem L, if there exists a polynomial equivalence relation $\sim$, a real constant $d \geqslant 1$, and an algorithm that, given $\sim$-equivalent strings $x_1, x_2, \ldots, x_t$, in time $\mathrm{poly}\left(t + \sum_{i=1}^{t} |x_i|\right)$ produces one instance $(y, k^\star)$ such that

- $(y, k^\star) \in L$ iff $x_i \in Q$ for at least one $i = 1, 2, \ldots, t$,
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- Constant d will be called the **dimension** of the weak cross-composition.

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Suppose an **NP**-hard problem Q admits a weak cross-composition into L with dimension d, and that L admits a polynomial compression with bitsize $\mathcal{O}(k^{d-\epsilon})$, for some $\epsilon > 0$. Then $\mathbf{NP} \subseteq \mathbf{coNP}/\mathrm{poly}$.

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- Example due to (Dell, Marx; 2012).

Multicolored Biclique

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Input: Bipartite graph H with bipartition $A \uplus B$, where $A = A_1 \uplus \ldots \uplus A_k$, $B = B_1 \uplus \ldots \uplus B_k$.

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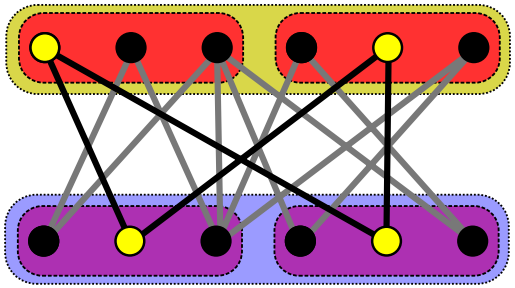

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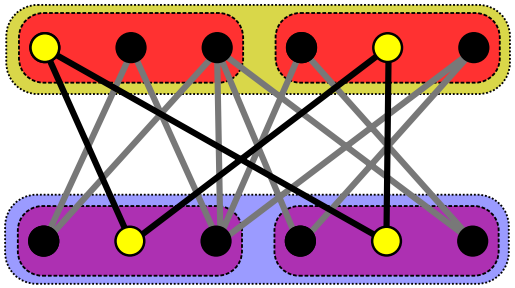

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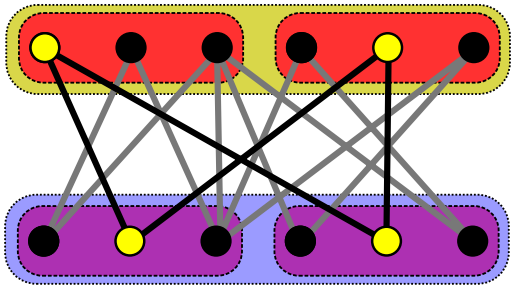

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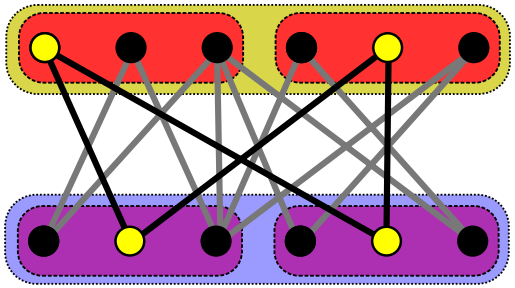

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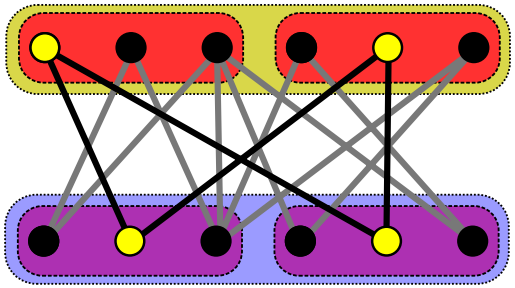
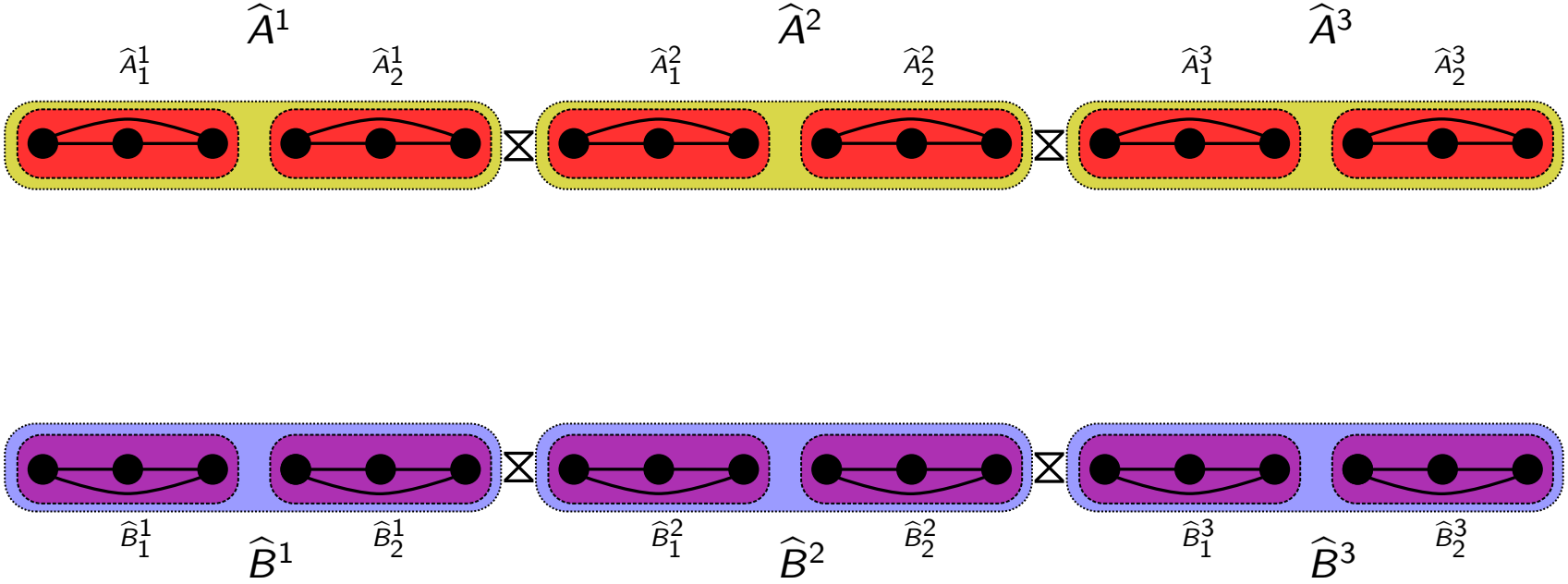
  - Thus all instances are on the same vertex set, partitioned in the same way; they are just given as different edge sets $E_1, \ldots, E_t$.



Composition

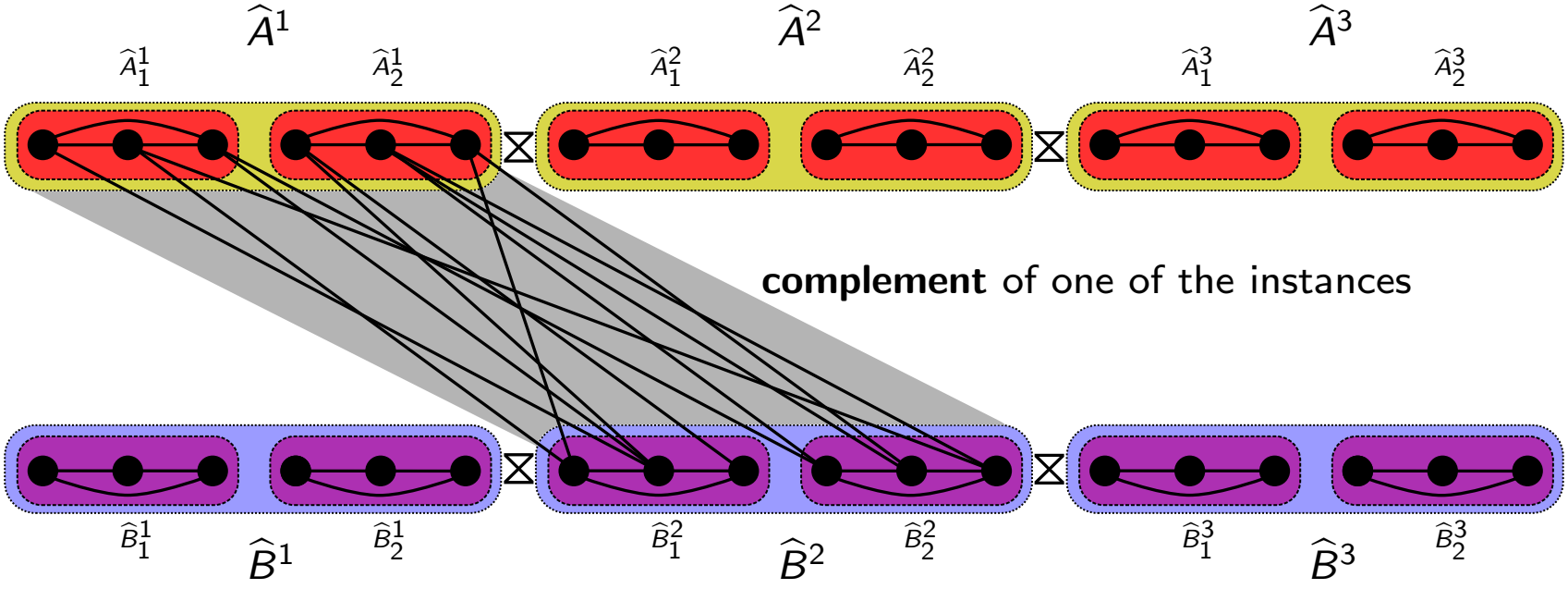
Create s copies of the left side and s copies of the right side.

$N = s \cdot 2kn \leqslant t^{1/2} \cdot \mathrm{poly}(\max_i |G_i|)$



Composition

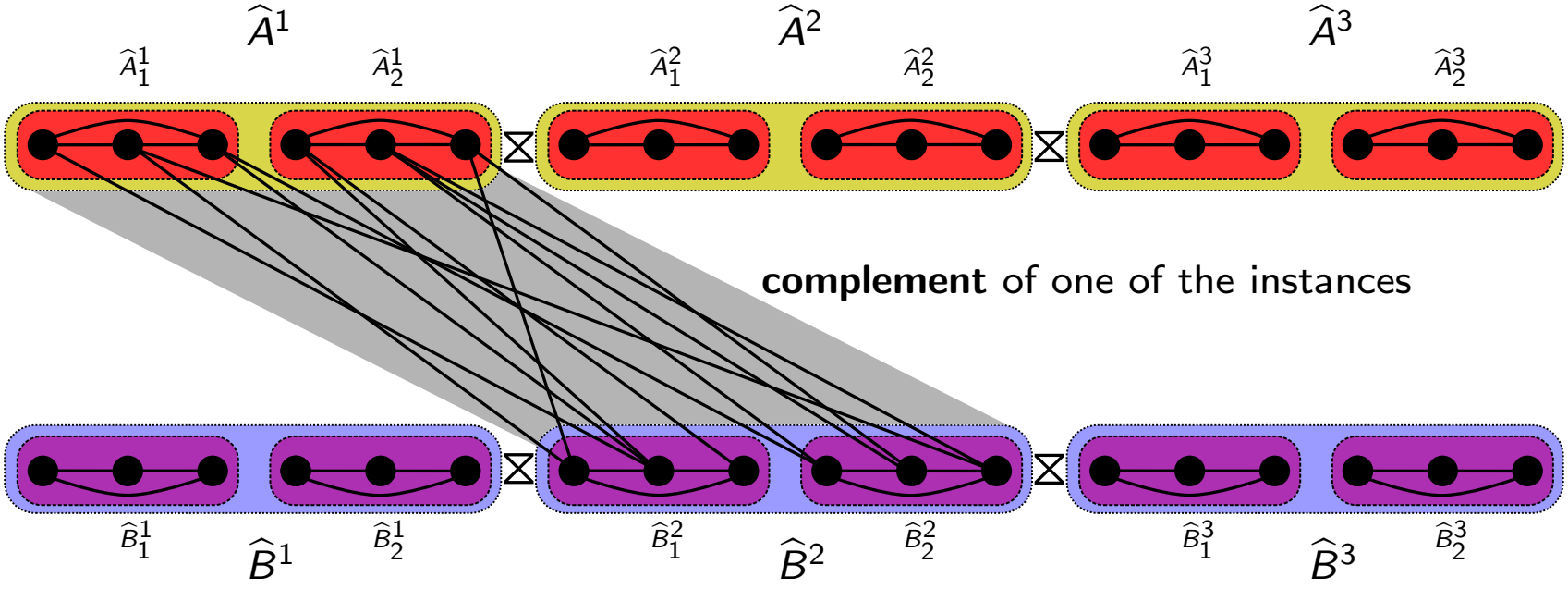
Embed s^2 instances into s^2 pairs of the sides.



Composition

Ask for an independent set of size $2k$;

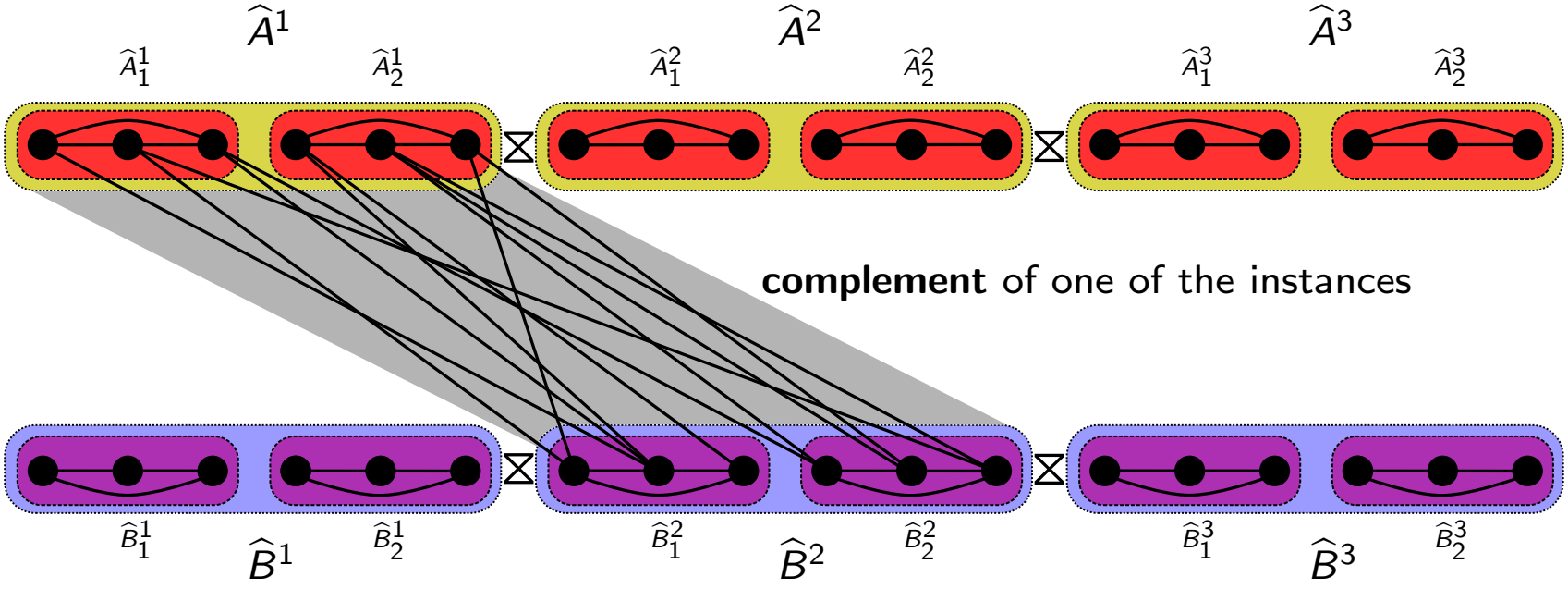
equivalently, a vertex cover of size $N - 2k$.



Composition

Edges on the sides ensure choosing one instance and respecting colors.

Edges originating in this instance ensure that the instance is solved.



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- Reduction from VC to FVS: add a degree-2 vertex to every edge, thus creating a triangle.
- **Cor**: FVS does not admit a $\mathcal{O}(k^{2-\epsilon})$ kernel.
- More $\mathcal{O}(k^{d-\epsilon})$ kernelization lower bounds for problems like Set Packing or Hitting Set for families of sets of size at most d.

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- **Answer**: Yes, but we do not understand it.

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