Structural sparsity and parameterized algorithms

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Recent Advances in Parameterized Complexity

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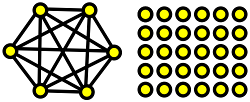

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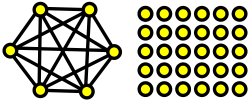

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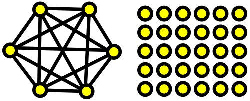

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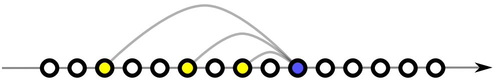

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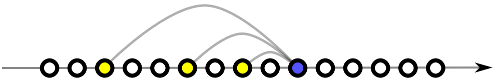

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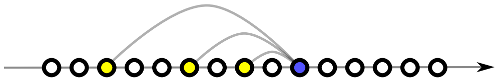

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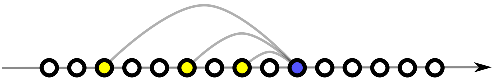

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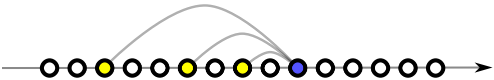

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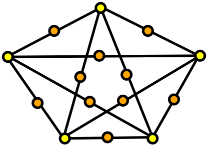

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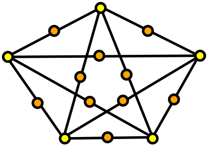

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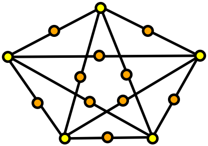

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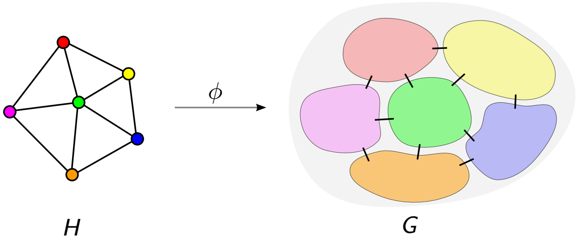

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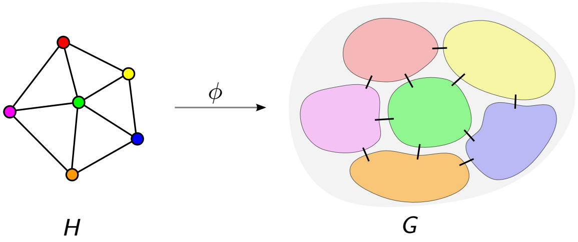

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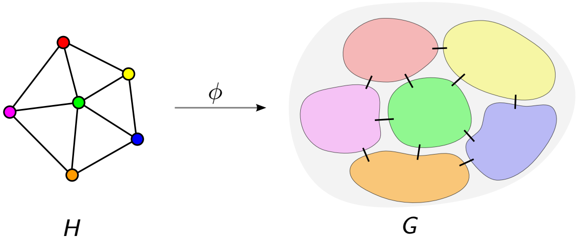

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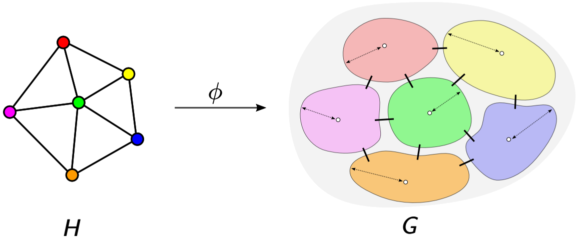

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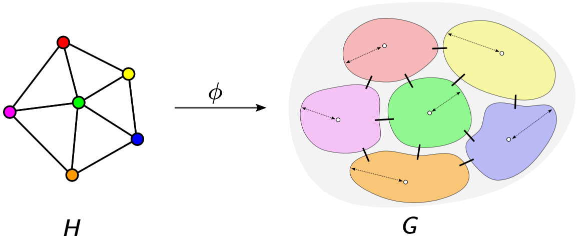

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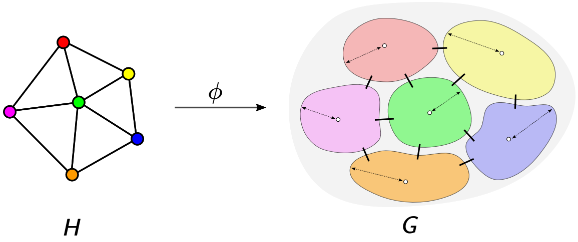
- **Idea**: Replace *subgraphs* with *shallow minors* in the def. of sparsity.



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Hierarchy of sparsity

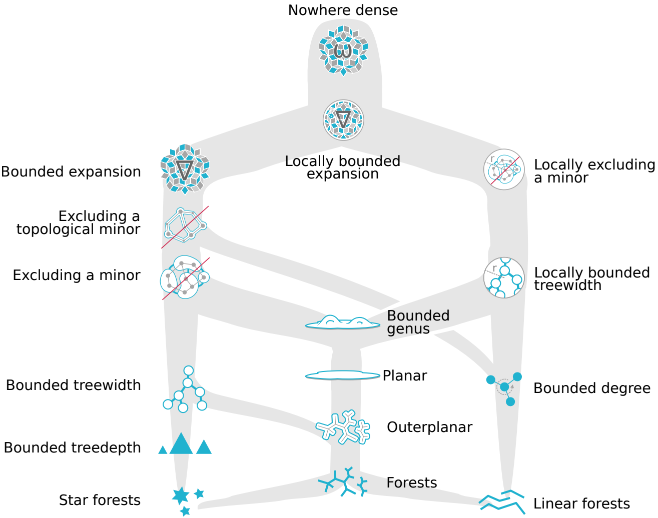

Figure by Felix Reidl

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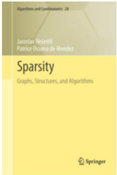

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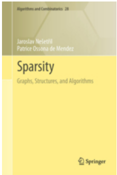

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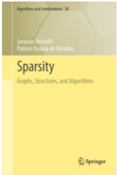
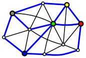


- **In summary**:
  - Bounded expansion and nowhere denseness are fundamental concepts that have multiple equivalent characterizations.
  - Each characterization yields a different viewpoint and a tool.
  - Applications for **combinatorics**, **algorithms**, and **logic**.
  - Nowhere denseness delimits tractability for many basic problems.
  - Toolbox seems much more suitable than using decomposition theorems for classes excluding a fixed (topological) minor.

Sparsity of shallow minors

Sparsity of shallow topological minors



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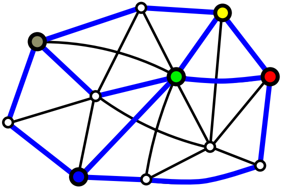
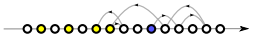


Characterizations

Generalized coloring numbers



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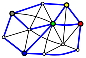

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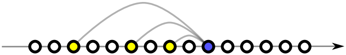
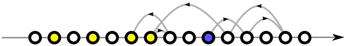


Degeneracy



Weak coloring number

Generalized coloring numbers

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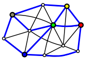

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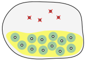
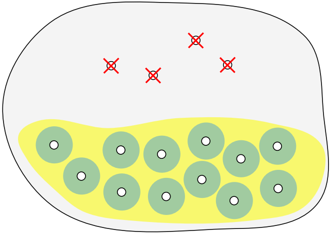
Uniform quasi-wideness





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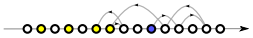

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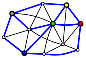

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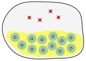
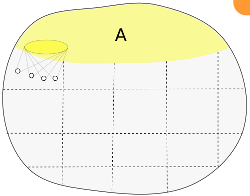
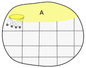




Neighborhood complexity



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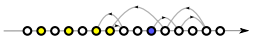

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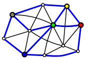

Fraternal augmentations

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Neighborhood covers

Low treedepth colorings

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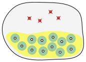

k-Helly property

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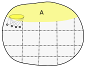

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- Inherently applies to problems with **local character**.

Local problems: algorithmics

r-Scattered Set

I Graph G and integer k

Q Is there $I \subseteq V(G)$ with $|I| = k$ s.t. r-balls around vertices of I are disjoint, equivalently vertices of I are pairwise at distance $> 2r$?

r-Dominating Set

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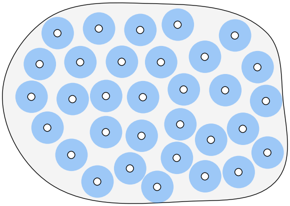
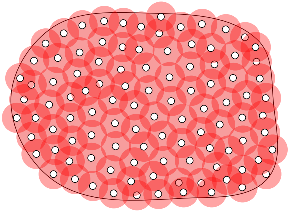
Q Is there $D \subseteq V(G)$ with $|D| = k$ s.t. every vertex of G is at distance $\leqslant r$ from some vertex of D?





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- Same story for r-Dominating Set and r-Independent Set.

Dichotomies

Theorem [Grohe et al., Dvořák et al.]

Let $\mathcal{C}$ be a monotone graph class (closed under taking subgraphs). Then:

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- Nowhere denseness **exactly** characterizes monotone classes where local problems are tractable from the parameterized viewpoint.

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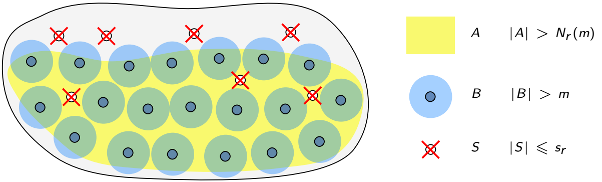

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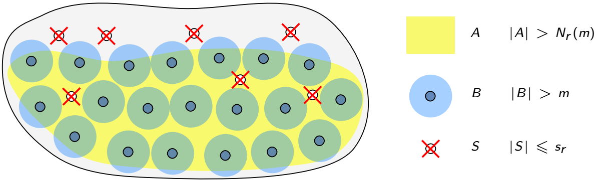

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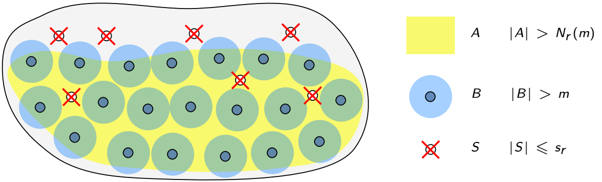

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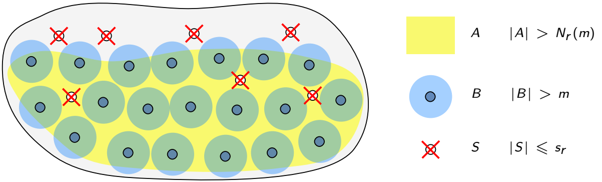

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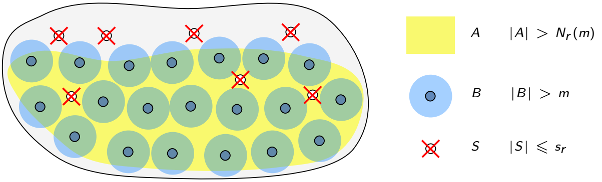
- B is r-scattered in $G - S$.



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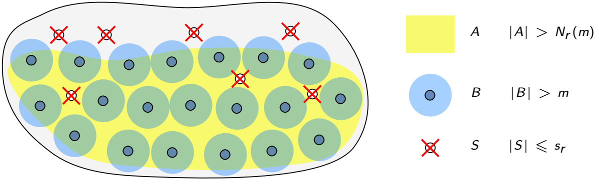

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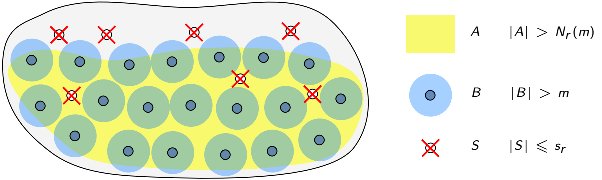

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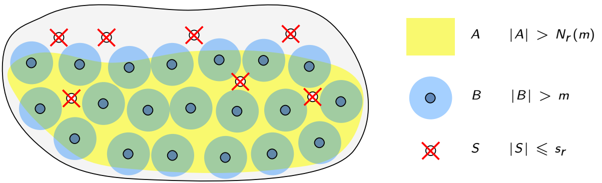

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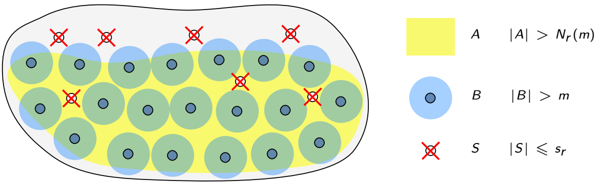

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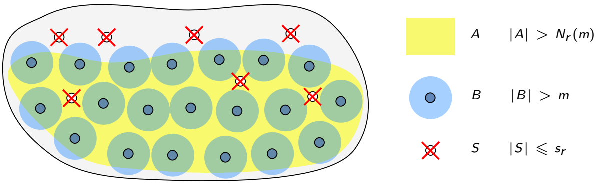

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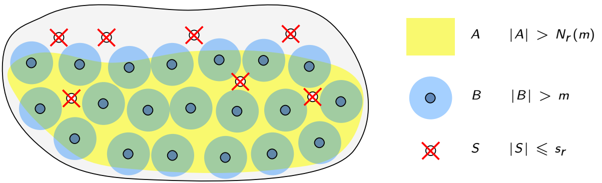

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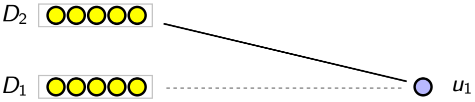

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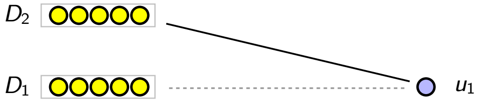

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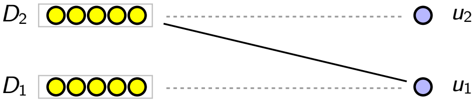

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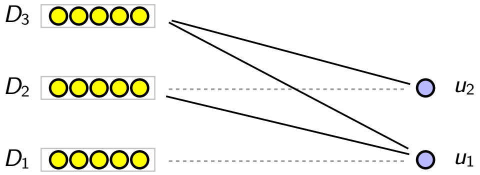

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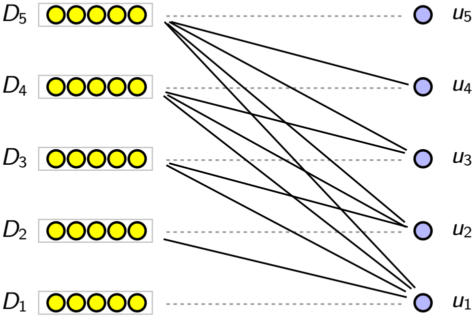

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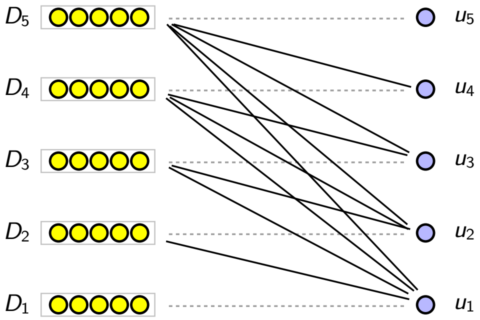

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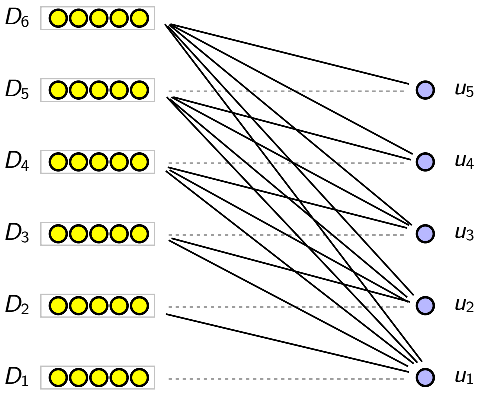

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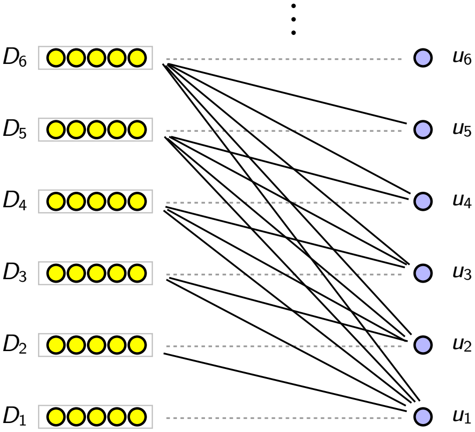

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- **Note**: The algorithm is oblivious of the class $\mathcal{C}$, it may be applied on any input graph G and parameter k.

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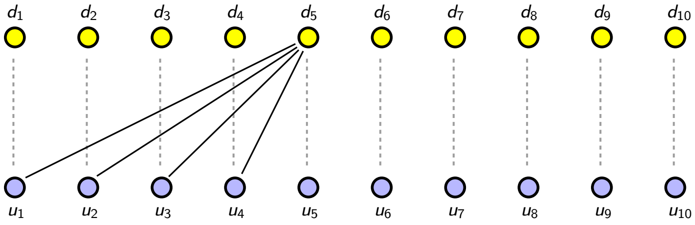

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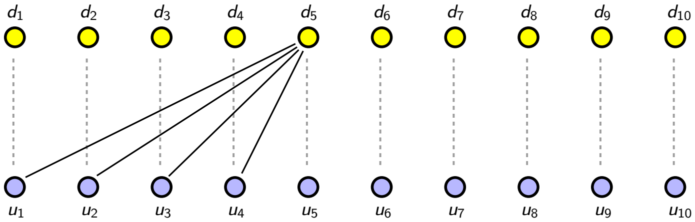

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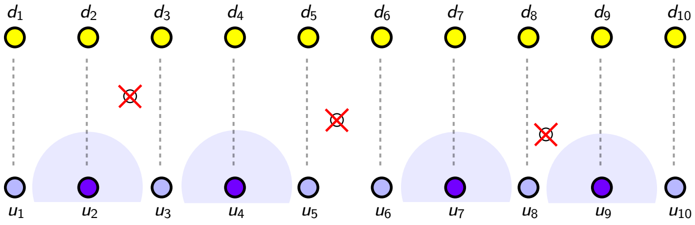

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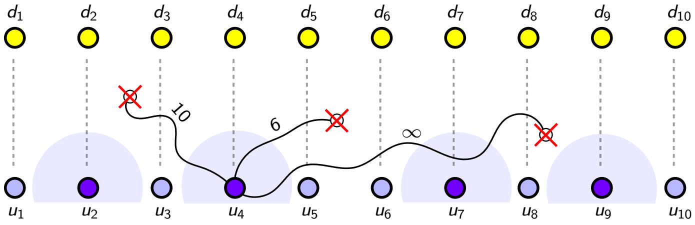

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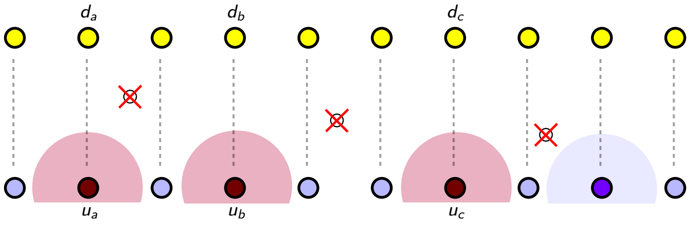

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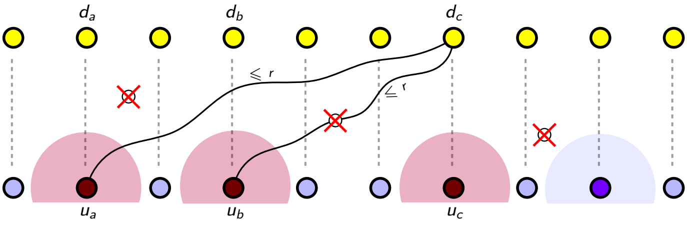

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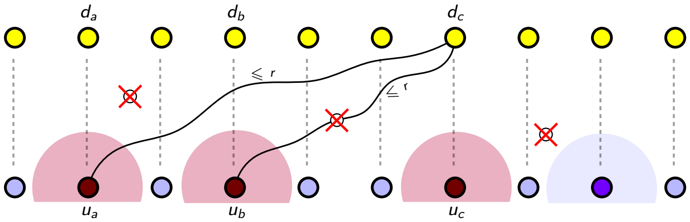

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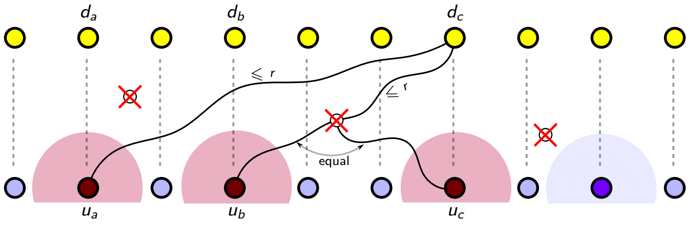

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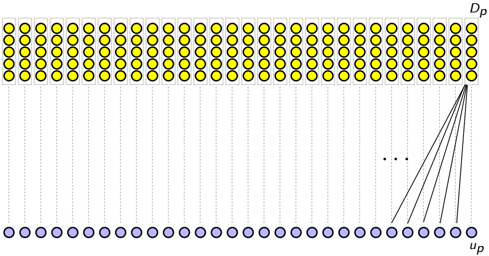

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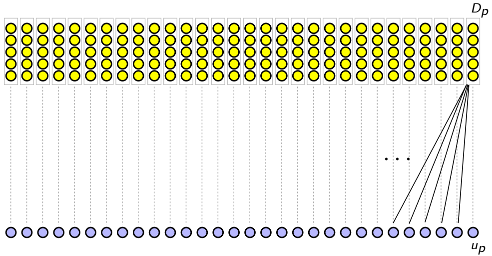

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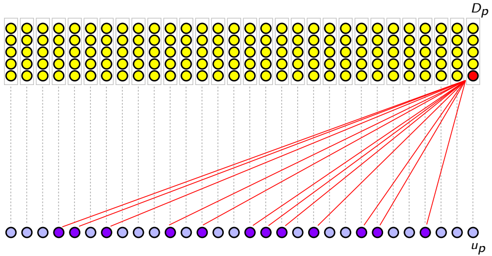

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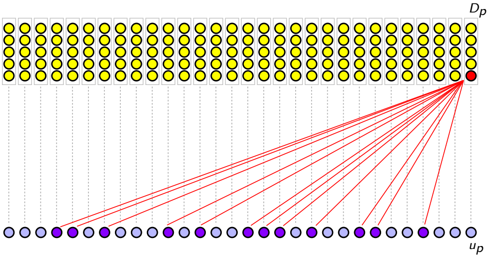

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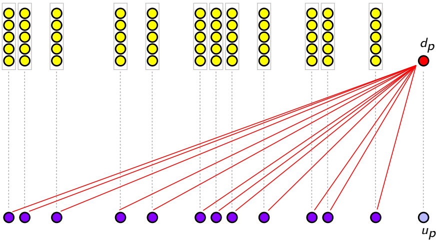

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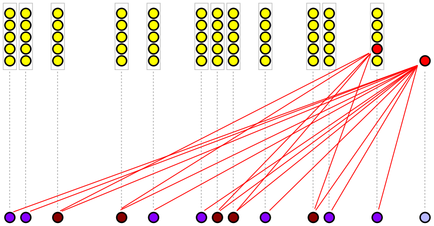

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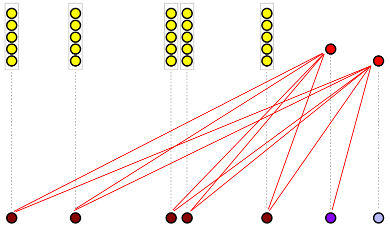

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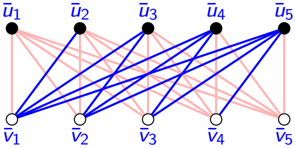

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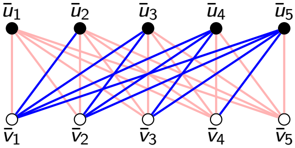

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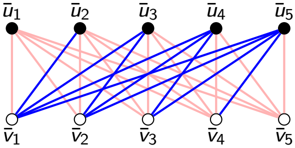

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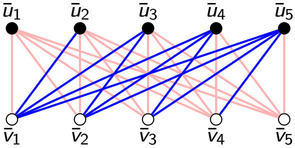

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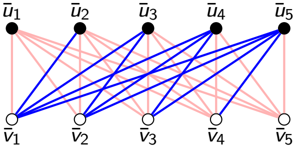

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- **Thank you for your attention!**